$$\underline{\int_{-2}}^{2} f = \sup_{P} L(f,P)$$

$$= \sup\{8, 10, 10.666\overline{6}, 11, 11.2, 11.333\overline{3}, \dots, 12\} = 12$$

$$\int_{-2}^{\overline{2}} f = \inf_{P} U(f,P)$$

$$= \inf\{16, 14, 13.333\overline{3}, 13, 12.8, 12.666\overline{6}, \dots, 12\} = 12$$

$$\Rightarrow \quad \int_{-2}^{2} f = \underline{\int_{-2}}^{2} f = \int_{-2}^{\overline{2}} f = 12$$

$\therefore$ f is Riemann integrable on $[-2,2]$, (or $f \in \mathcal{R}[-2,2]$).

Example (2.8): Let $f: [0,1] \to R$ defined as $f(x) = x^2$, prove that f is Riemann integrable on $[0,1]$.

Solution:

Let $P_n = \left\{\left[0, \frac{1}{n}\right], \left[\frac{1}{n}, \frac{2}{n}\right], \left[\frac{2}{n}, \frac{3}{n}\right], \dots, \left[\frac{i-1}{n}, \frac{i}{n}\right], \dots, \left[1 - \frac{1}{n}, 1\right]\right\}$

We have

$\delta x_i = \frac{1}{n}$, $\forall\ i = 1,2,3, \dots, n.$

and

$m_i = \inf\left\{f(x): x \in \left[\frac{i-1}{n}, \frac{i}{n}\right]\right\} = \frac{(i-1)^2}{n^2}, \quad \forall\ i = 1,2,3, \dots, n.$

$M_i = \sup\left\{f(x): x \in \left[\frac{i-1}{n}, \frac{i}{n}\right]\right\} = \frac{i^2}{n^2}, \quad \forall\ i = 1,2,3, \dots, n.$

It follows that

$$L(f, P_n) = \sum_{i=1}^{n} m_i \delta x_i = m_1\delta x_1 + m_2\delta x_2 + m_3\delta x_3 + \cdots + m_n\delta x_n$$

$$= \frac{0^2}{n^2} \times \frac{1}{n} + \frac{1^2}{n^2} \times \frac{1}{n} + \frac{2^2}{n^2} \times \frac{1}{n} + \cdots + \frac{(n-1)^2}{n^2} \times \frac{1}{n}$$

$$= 0 + \frac{1^2}{n^3} + \frac{2^2}{n^3} + \cdots + \frac{(n-1)^2}{n^3}$$

$$= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + (n-1)^2)$$

$$= \frac{1}{6n^3}(n(n+1)(2n+1) - n^2)$$

$$= \frac{1}{6n^3}(2n^3 + 2n^2 + n)$$

$$= \frac{1}{6n^3}n^3(2 + \frac{2}{n} + \frac{1}{n^2}) = \frac{1}{6}(2 - \frac{3}{n} + \frac{1}{n^2})$$

$$U(f, P_n) = \sum_{i=1}^{n} M_i \delta x_i = M_1 \delta x_1 + M_2 \delta x_2 + M_3 \delta x_3 + \cdots + M_n \delta x_n$$

$$= \frac{1^2}{n^2} \times \frac{1}{n} + \frac{2^2}{n^2} \times \frac{1}{n} + \frac{3^2}{n^2} \times \frac{1}{n} + \cdots + \frac{n^2}{n^2} \times \frac{1}{n}$$

$$= \frac{1^2}{n^3} + \frac{2^2}{n^3} + \frac{3^2}{n^3} + \cdots + \frac{n^2}{n^3}$$

$$= \frac{1}{n^3}(1^2 + 2^2 + 3^2 + \cdots + n^2)$$

$$= \frac{1}{6n^3}n(n+1)(2n+1)$$

$$= \frac{1}{6n^3}(2n^3 + 3n^2 + n)$$

$$= \frac{1}{6n^3}n^3(2 + \frac{3}{n} + \frac{1}{n^2}) = \frac{1}{6}(2 + \frac{3}{n} + \frac{1}{n^2})$$

We get

$$\int_{\underline{0}}^{1} f = \sup_{P_n} L(f, P_n) = \lim_{n\to\infty} L(f, P_n)$$

$$= \lim_{n\to\infty} \left\{\frac{1}{6}(2 + \frac{2}{n} + \frac{1}{n^2})\right\} = \frac{1}{3}$$

$$\int_{0}^{\overline{1}} f = \inf_{P_n} U(f, P_n) = \lim_{n\to\infty} U(f, P_n)$$

$$= \lim_{n\to\infty} \left\{\frac{1}{6}(2 + \frac{3}{n} + \frac{1}{n^2})\right\} = \frac{1}{3}$$

$$\Rightarrow \quad \int_0^1 f = \int_{\underline{0}}^{1} f = \int_0^{\overline{1}} f = \frac{1}{3}$$

$\therefore$ f is Riemann integrable on [0,1], (or $f \in \mathcal{R}[0,1]$).

Exercises (2.1): (Homework)

(1) Find the upper and lower Riemann integrals for $f(x) = x^3$ on the interval $[0, b]$.

(2) Let $f: [0,2] \to R$ defined as $f(x) = \begin{cases} 5 & if \quad x < 1 \\ 3 & if \quad x \geq 1 \end{cases}$. Is f Riemann integrable on $[0,2]$.

Basic Properties of Riemann Integration

Theorem (2.2): Let f be a <u>bounded</u> function on an interval $[a, b]$. Then:

(a) The lower Riemann integral $\underline{\int_a^b} f$ and the upper Riemann integral $\overline{\int_a^b} f$ both exist;

(b) $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

Proof:

(a) Since f is bounded on [a, b],

$\Rightarrow \exists$ some number M such that $|f(x)| \leq M$ on $[a, b]$

$\Rightarrow f(x) \leq M$, for $x \in [a, b]$

$\Rightarrow f(x) \leq M$, for $x \in [x_{i-1}, x_i]$, $\forall P = \{[x_{i-1}, x_i]: 1 \leq i \leq n\}$ of $[a, b]$

We have

$$m_i = \inf_{[x_{i-1}, x_i]} f(x) \leq M$$

$$\Rightarrow L(f, P) = \sum_{i=1}^{n} m_i \delta x_i \leq \sum_{i=1}^{n} M \, \delta x_i$$

$$= M \sum_{i=1}^{n} \delta x_i = M(b - a)$$

Since $L(f, P) \leq M(b - a)$

$\Rightarrow \ \sup\limits_{P} L(f,P)$ exist

$\therefore \ \underline{\int_a^b} f$ exist

By the same way we can prove the existence of $\overline{\int_a^b} f$.

(b) From Theorem (2.1), we have

$L(f,P) \le U(f,P)$

$\Rightarrow \ \sup\limits_{P} L(f,P) \le \inf\limits_{P} U(f,P)$

$$\Rightarrow \ \underline{\int_a^b} f \le \overline{\int_a^b} f.$$

Theorem (2.3): (Riemann's Criterion for Integrability)

Let f be a bounded function on an interval $[a,b]$. Then f is Riemann integrable on $[a,b]$ if and only if for each positive number ε, there is a partition P of $[a,b]$ for which $U(f,P) - L(f,P) < \varepsilon$.

Theorem (2.4): Let f be a bounded function on $[a,b]$. If f continuous on $[a,b]$. Then f is Riemann integrable on $[a,b]$.

Proof:

Since f is continuous on closed interval $[a,b]$

$\Rightarrow f$ is uniformly continuous on $[a,b]$.

$\Rightarrow \forall\ \varepsilon > 0, \frac{\varepsilon}{2(b-a)} > 0, \exists\ \delta > 0$ such that

$|x-y| < \delta \quad \Rightarrow \quad |f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}, \quad \forall\ x,y \in [a,b]$ ...(1)

Now, let $P = \{[x_{i-1}, x_i]\}_{i=1}^{n}$ be a partition of $[a,b]$ with mesh $\|P\| < \delta$. Then We have

$|x-y| \le x_i - x_{i-1}$ (for each i, $\forall\ x,y \in [x_{i-1}, x_i]$)