Theorem:- let R be a relation in A. Then R is symmetric iff $R^{-1}=R$.

Proof:- suppose that R is symmetric

let $(a,b)\in R^{-1} \xRightarrow{\text{تعريف } R^{-1}} (b,a)\in R \xRightarrow{\text{متناظرة}} (a,b)\in R$

$\Rightarrow R^{-1}\subseteq R \quad \cdots ①$

let $(x,y)\in R$ $\because R$ is symmetric

$\Rightarrow (y,x)\in R \Rightarrow (x,y)\in R^{-1}$

$\Rightarrow R\subseteq R^{-1} \quad \cdots ②$

from ① and ② we get $R^{-1}=R$.

$\Leftarrow$ assume that $R^{-1}=R$.

let $(a,b)\in R \xRightarrow[R^{-1}=R]{} (a,b)\in R^{-1} \xRightarrow[\text{تعريف } R^{-1}]{} (b,a)\in R$

$\therefore R$ is symmetric

EX.:-

① $R=\{(x,y)\in Z\times Z : \frac{x+y}{3}\}$. Is R equivalence relation? why?

② $R=\{(x,y)\in Z\times Z : \frac{x+y}{4}\}$. Is R equivalence relation? why?

$\frac{\overset{5}{x}+\overset{1}{y}}{3}, \frac{\overset{1}{y}+\overset{8}{z}}{3} \Rightarrow \frac{\overset{5}{x}+\overset{8}{z}}{3} \notin R$ not transitive.

③ $R=\{(x,y)\in Z\times Z : \frac{x+y}{11}\}$. Is R equivalence relation? why?

$\frac{\overset{9}{x}+\overset{2}{y}}{11}, \frac{\overset{2}{y}+\overset{20}{z}}{11} \Rightarrow \frac{\overset{9}{x}+\overset{20}{z}}{11} \notin R$ not transitive.

Th. let R_1 and R_2 are symmetric relation, then ① $R_1 \cap R_2$ symmetric ② $R_1 \cup R$ is symm.

Proof:-

① let $(x,y) \in R_1 \cap R_2 \Rightarrow (x,y) \in R_1$ and $(x,y) \in R_2$

since R_1 and R_2 symmetric, then

$(y,x) \in R_1$ and $(y,x) \in R_2 \Rightarrow (y,x) \in R_1 \cap R_2$

$\therefore R_1 \cap R_2$ is symmetric.

② let $(x,y) \in R_1 \cup R_2 \Rightarrow (x,y) \in R_1$ or $(x,y) \in R_2$

since R_1 and R_2 are symmetric.

$\Rightarrow (y,x) \in R_1$ or $(y,x) \in R_2 \Rightarrow (y,x) \in R_1 \cup R_2$

$\therefore R_1 \cup R_2$ is symmetric.

Th. let R_1 and R_2 are transitive relation, then $R_1 \cap R_2$ is transitive.

(x,y) and $(y,z) \in R$
$\Rightarrow (x,z) \in R$

Proof:- let $(x,y) \in R_1 \cap R_2$ and $(y,z) \in R_1 \cap R_2$

$\Rightarrow (x,y) \in R_1$ and $(x,y) \in R_2$ and $(y,z) \in R_1$ and $(y,z) \in R_2$

$\Rightarrow [(x,y) \in R_1$ and $(y,z) \in R_1]$ and $[(x,y) \in R_2$ and $(y,z) \in R_2]$

$\Rightarrow (x,z) \in R_1$ (R_1 is transitive) and

$(x,z) \in R_2$ (R_2 is transitive)

$\therefore (x,z) \in R_1 \cap R_2$, $R_1 \cap R_2$ is transitive.

Q. :- If R_1 and R_2 are transitive. Is $R_1 \cup R_2$ transitive? why?

مهم

اذا كانت R_1 و R_2 متعديان فهل ان $R_1 \cup R_2$ متعدية

الجواب كلا؟ مثال على ذلك

let $A = \{0, 1, 2, 3\}$

$R_1 = \{(0,0), (1,1), (2,2), (3,3), (1,3), (3,1)\}$

$R_2 = \{(0,0), (1,1), (2,2), (3,3), (1,2), (2,1)\}$

$R_1 \cup R_2 = \{(0,0), (1,1), (2,2), (3,3), (1,3), (3,1), (1,2), (2,1)\}$

R_1 is transitive, R_2 is transitive

$(3,1)$ and $(1,2) \in R_1 \cup R_2$ but $(3,2) \notin R_1 \cup R_2$

$\therefore R_1 \cup R_2$ is not transitive.

Note :- ① If R_1 and R_2 are equivalence relation then $R_1 \cap R_2$ is equivalence relation -

H.W

نفس المثال اعلاه ② $R_1 \cup R_2$ is not equivalence relation.

Ex.:- let $R = \{ ((a,b),(c,d)) \in (\mathbb{R}\times\mathbb{R})\times(\mathbb{R}\times\mathbb{R}) : a+b = c+d \}$

prove that R is equivalence.

Sol.:- let $(a,b),(c,d) \in R$

$\Rightarrow a+b = c+d = a+b$

$\Rightarrow ((a,b),(a,b)) \in R$

$\therefore R$ is reflexive

let $((a,b),(c,d)) \in R$ 　　 يعكس الزوج المرتب كله

$\Rightarrow a+b = c+d \Rightarrow c+d = a+b$

$\Rightarrow ((c,d),(a,b)) \in R$

$\therefore R$ is symm.

let $(a,b),(c,d) \in R$ and $(c,d),(e,f) \in R$

$\Rightarrow a+b = c+d \ \wedge \ c+d = e+f$

$\Rightarrow a+b = e+f$

$\Rightarrow ((a,b),(e,f)) \in R$

$\therefore R$ is transitive

Def.:- Let A be a set and R is equivalence relation defined on A, if $a \in A$, then the (equivalence classes) of a modulo R is the set $[a]$ defined as follows:-

التكافؤ صفوف

$$[a] = \{x \in A : (x, a) \in R\}$$

or $$[a] = \{x \in A : xRa\}$$

(شرط ان تكون R علاقة تكافؤ)

Ex.:- let $A = \{x, y, z\}$ and

$$R = I_A \cup \{(x,y), (y,x), (x,z), (z,x), (y,z), (z,y)\}$$

معناها هذه العلاقة متكافئة

Find $[x], [y], [z]$

$[x] = \{x, y, z\}$, $[y] = \{y, x, z\}$, $[z] = \{z, x, y\}$.

له كنده علاقة مع اي عنصر

Ex.:- Let $R = \{(x,y) \in Z \times Z : \frac{x-y}{3}\}$ is a relation find $[0], [1], [-1], [4]$.

Sol.:- R is equivalence relation. H.W

$$R = \{(x,y) \in Z \times Z : x-y/3\}$$

$$= \{(x,y) \in Z \times Z, \exists k \in Z : x-y = 3k\}$$

$$[0] = \{x \in Z : \exists k \in Z, xR0\}$$

$$= \{x \in Z : \exists k \in Z, x-0 = 3k\}$$

$$= \{x : \exists k \in Z, x = 3k\}$$

$$= \{0, 3, -3, 6, -6, \ldots\}$$

(23)

$[1] = \{x \in \mathbb{Z}, x R 1\}$

$= \{x : \exists k \in \mathbb{Z}, x - 1 = 3k\}$

$= \{x : \exists k \in \mathbb{Z}, x = 3k + 1\}$

$= \{1, 4, -2, 7, \ldots\}$

$[4] = \{x \in \mathbb{Z}, x R 4\}$

$= \{x : \exists k \in \mathbb{Z}, x - 4 = 3k\}$

$= \{x : \exists k \in \mathbb{Z}, x = 3k + 4\}$

$= \{4, 7, 1, \ldots\}$

$[-1] = \{x : \exists k \in \mathbb{Z}, \underbrace{x - (-1)}_{x+1} = 3k\}$

$= \{x : \exists k \in \mathbb{Z}, x = 3k - 1\}$

$= \{-1, 2, -4, 7, \ldots\}$

Ex. :- Let $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : a - b = \text{even}\}$. Find $[1], [3]$

Sol. :- $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : a - b = 2k, k \in \mathbb{Z}\}$

$= \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : a = 2k + b, k \in \mathbb{Z}\}$

اولاً نختبر R يجب ان تكون علاقة تكافؤ

R is equivalence relation.

$[1] = \{a : \exists k \in \mathbb{Z}, a - 1 = 2k\}$

$= \{a : \exists k \in \mathbb{Z}, a = 2k + 1\}$

$= \{1, 3, -1, 5, \ldots\}$

$[3] = \{a : \exists k \in \mathbb{Z}, a = 2k + 3\}$

$= \{3, 5, 1, 7, -1, \ldots\}$

$\therefore [1] = [3]$.

Theorem:- Let R be an equivalence relation in A. (24)

Then 1- $a \in [a]$, $\forall a \in R$.

2- if $b \in [a]$ and $a \in [b]$, then $[a] = [b]$.

3- $[a] = [b]$ iff $(a,b) \in R$.

4- if $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.

Proof(1):- $[a] = \{x \in A : (x,a) \in R\}$

$\because R$ is equivalence relation in A. Then R is reflexive

$\Rightarrow \forall a \in A$, $(a,a) \in R \Rightarrow a \in [a]$ (by def. of $[a]$).

Proof(2):- let $x \in [a] \Rightarrow (x,a) \in R$

$\because b \in [a] \Rightarrow (b,a) \in R$

since R is symmetric $\Rightarrow (a,b) \in R$

$\because (x,a) \in R$ and $(a,b) \in R$

$\Rightarrow (x,b) \in R$ (R is transitive)

$\Rightarrow x \in [b]$

$\therefore [a] \subseteq [b]$ ---- ①

Now, let $y \in [b] \Rightarrow (y,b) \in R$

$a \in [b] \Rightarrow (a,b) \in R$

$\because R$ is symmetric $\Rightarrow (b,a) \in R$

$\therefore (y,b) \in R$ and $(b,a) \in R$

$\Rightarrow (y,a) \in R$ (R is transitive)

$\Rightarrow y \in [a]$

$\therefore [b] \subseteq [a]$ --- ②

From ① and ② we get $[a] = [b]$.

(3) suppose that $[a]=[b]$

$\because a \in [a]$ from (1)

$\because [a]=[b] \Rightarrow a \in [b] \Rightarrow (a,b) \in R$

$\Leftarrow$ assume that $(a,b) \in R$, to prove $[a]=[b]$

let $x \in [a] \Rightarrow (x,a) \in R$

$\because (a,b) \in R \Rightarrow (x,b) \in R$ (since R is tra.)

$\Rightarrow x \in [b] \Longrightarrow [a] \subseteq [b]$ ---(1)

Now, let $y \in [b] \Rightarrow (y,b) \in R$ $\therefore (a,b) \in R$

$\Rightarrow (b,a) \in R$ since (R is symm.)

$\Rightarrow (y,b)$ and $(b,a) \in R$

$\Rightarrow (y,a) \in R$ since (R is tra.) $\Rightarrow y \in [a]$

$\Rightarrow [b] \subseteq [a]$ ---(2)

from (1) and (2) we get $[a]=[b]$.

proof (4):- $\because [a] \cap [b] \neq \emptyset$

$\Rightarrow z \in [a] \cap [b]$

$\Rightarrow \exists z \in [a]$ and $z \in [b]$

$\Rightarrow (z,a) \in R$ and $(z,b) \in R$

$\Rightarrow (a,z) \in R$ and $(z,b) \in R$ (since R is sym.)

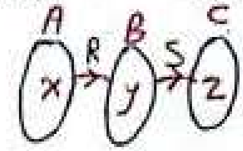
$\Rightarrow (a,b) \in R$ since (R is tra.)

from (3) we get

$[a]=[b]$.

Def.:- If R is a relation from A into B and S is a relation from B into C, then تركيب العلاقات

$$S \circ R = \{ (x,z) \in A \times C ,\ \exists\, y \in B \text{ s.t } (x,y) \in R \text{ and } (y,z) \in S \}$$



Note:- $R \circ S \neq S \circ R$

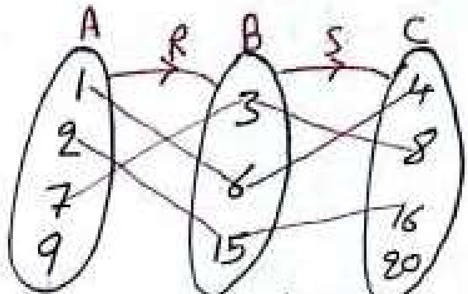
Ex.:- let $A = \{1,2,7,9\}$, $B = \{3,6,15\}$, $C = \{4,8,16,20\}$

$R = \{(1,6), (2,15), (7,3)\}$

$S = \{(3,8), (15,16), (6,4)\}$

$S \circ R = \{(1,4), (2,16), (7,8)\}$

$R \circ S = \varnothing$



هذه غير دالة لان 9 ليس لها صورة

Theorem:- Let R be a relation defined on A. Then

$$R \circ I_A = I_A \circ R = R.$$

Proof:-

iden. relation

let $(x,z) \in R \circ I_A$ $\qquad (x,y) \in I_A \wedge (y,z) \in R$

$\Rightarrow \exists\, y \in A \text{ s.t } (x,y) \in I_A \text{ and } (y,z) \in R$

$\because (x,y) \in I_A \Rightarrow x = y \Rightarrow (x,x) \in I_A \text{ and } (y,z) \in R$

$\Rightarrow (x,z) \in R \Rightarrow R \circ I_A \subseteq R$ --- ①

Now, let $(x,z) \in R \Rightarrow (x,x) \in I_A \text{ and } (x,z) \in R$

$\Rightarrow (x,z) \in R \circ I_A$

$\Rightarrow R \subseteq R \circ I_A$ --- ②

From ① and ② we get $R \circ I_A = R$.

Th. let R, S and T are relations defined on the set A. Then

① $(T \circ S) \circ R = T \circ (S \circ R)$.

② $(S \cup T) \circ R = (S \circ R) \cup (T \circ R)$.

③ $(S \cap T) \circ R = (S \circ R) \cap (T \circ R)$.

④ If $R \subseteq S$, then $T \circ R \subseteq T \circ S$, $(R \circ T \subseteq S \circ T)$.

⑤ $(S \circ R) \cap T = \emptyset$ (then) iff $(T \circ R^{-1}) \cap S = \emptyset$.

⑥ $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$ and $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$.

① $(T \circ S) \circ R = T \circ (S \circ R)$

proof:- let $(x,z) \in (T \circ S) \circ R$

$\Leftrightarrow \exists y \in A$ s.t $(x,y) \in R$ and $(y,z) \in (T \circ S)$

$\Leftrightarrow \exists y \in A$ s.t $(x,y) \in R$ and $\exists l \in A$ s.t $(y,l) \in S$ and $(l,z) \in T$

$\Leftrightarrow \exists y \in A$ and $l \in A$ s.t $(x,y) \in R$ and $(y,l) \in S$ and $(l,z) \in T$

$\Leftrightarrow \exists l \in A$ s.t $(x,l) \in (S \circ R)$ and $(l,z) \in T$

$\Leftrightarrow (x,z) \in T \circ (S \circ R)$

$\Leftrightarrow (T \circ S) \circ R \subseteq T \circ (S \circ R)$ ----①

وبعكس الخطوات نحصل على

$T \circ (S \circ R) \subseteq (T \circ S) \circ R$ ----②

From ① and ② we get

$(T \circ S) \circ R = T \circ (S \circ R)$.

② $(S \cup T) \circ R = (S \circ R) \cup (T \circ R)$.

proof:- let $(x,z) \in (S \cup T) \circ R$

$\Rightarrow \exists y \in A \ni (x,y) \in R$ and $(y,z) \in S \cup T$

$\Rightarrow \exists y \in A \ni (x,y) \in R$ and $(y,z) \in S$ or $(y,z) \in T$

$\Rightarrow [(x,y) \in R$ and $(y,z) \in S]$ or $[(x,y) \in R$ and $(y,z) \in T]$

$\Rightarrow (x,z) \in S \circ R$ or $(x,z) \in T \circ R$

$\Rightarrow (x,z) \in (S \circ R) \cup (T \circ R)$.

$\Rightarrow (S \cup T) \circ R \in (S \circ R) \cup (T \circ R)$ ---- ①

$\Leftarrow$ H.W

③ $(S \cap T) \circ R = (S \circ R) \cap (T \circ R)$.

proof:- let $(x,z) \in (S \cap T) \circ R$

$\Rightarrow \exists y \in A \ni (x,y) \in R$ and $(y,z) \in S \cap T$

$\Rightarrow \exists y \in A \ni (x,y) \in R$ and $(y,z) \in S$ and $(y,z) \in T$

$\Rightarrow [(x,y) \in R$ and $(y,z) \in S]$ and $[(x,y) \in R$ and $(y,z) \in T]$

$\Rightarrow (x,z) \in (S \circ R)$ and $(x,z) \in (T \circ R)$

$\Rightarrow (x,z) \in (S \circ R) \cap (T \circ R)$

$\Rightarrow (S \cap T) \circ R \subseteq (S \circ R) \cap (T \circ R)$ — ①

$\Leftarrow$ H.W

④ If $R \subseteq S$, then $T \circ R \subseteq T \circ S$, $(R \circ T \subseteq S \circ T)$.

Proof:- let $(x,z) \in T \circ R$

$\Rightarrow \exists y \in A \ni (x,y) \in R$ and $(y,z) \in T$

$\because R \subseteq S \Rightarrow (x,y) \in S$ and $(y,z) \in T$

$\Rightarrow (x,z) \in T \circ S$.

$\Rightarrow T \circ R \subseteq T \circ S$.

let $(x,z) \in R \circ T$

$\Rightarrow \exists y \in A \Rightarrow (x,y) \in T$ and $(y,z) \in R$

$\because R \subseteq S \Rightarrow (x,y) \in T$ and $(y,z) \in S$

$\Rightarrow (x,z) \in S \circ T$

$\Rightarrow R \circ T \subseteq S \circ T$.

⑤ $(S \circ R) \cap T = \emptyset$, (then) iff $(T \circ R^{-1}) \cap S = \emptyset$

Proof:- assume that $(T \circ R^{-1}) \cap S = \emptyset$

if $(S \circ R) \cap T \neq \emptyset$

let $(x,z) \in (S \circ R) \cap T$

$\Rightarrow (x,z) \in (S \circ R)$ and $(x,z) \in T$

$\Rightarrow \exists y \in A \ni (x,y) \in R$ and $(y,z) \in S$ and $(x,z) \in T$

$\Rightarrow \exists y \in A \ni (y,x) \in R^{-1}$ and $(y,z) \in S$ and $(x,z) \in T$

$\Rightarrow \exists y \in A \ni (y,x) \in R^{-1}$ and $(x,z) \in T$ and $(y,z) \in S$

$\Rightarrow (y,z) \in T \circ R^{-1}$ and $(y,z) \in S$

$\Rightarrow (y,z) \in (T \circ R^{-1}) \cap S$

$\Rightarrow (T \circ R^{-1}) \cap S \neq \emptyset$ and that C! $\therefore (S \circ R) \cap T = \emptyset$

$\Leftarrow$ suppose that $(S \circ R) \cap T = \emptyset$

if $(T \circ R^{-1}) \cap S \neq \emptyset$

let $(x,z) \in (T \circ R^{-1}) \cap S$

$\Rightarrow (x,z) \in T \circ R^{-1}$ and $(x,z) \in S$

$\Rightarrow \exists y \in A \ni (x,y) \in R^{-1}$ and $(y,z) \in T$ and $(x,z) \in S$

$\Rightarrow \exists y \in A \ni (y,x) \in R$ and $(y,z) \in T$ and $(x,z) \in S$

$\Rightarrow (y,x) \in R$ and $(x,z) \in S$ and $(y,z) \in T$

$\Rightarrow (y,z) \in (S \circ R)$ and $(y,z) \in T$

$\Rightarrow (y,z) \in (S \circ R) \cap T \neq \emptyset$ C!

$\therefore (T \circ R^{-1}) \cap S = \emptyset$.

⑥ $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$ and $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$. H.W

Proof:- let $(x,z) \in (S \circ R)^{-1}$

$\Rightarrow (z,x) \in (S \circ R)$

$\Rightarrow \exists y \in A \ni (z,y) \in R$ and $(y,x) \in S$

$\Rightarrow \exists y \in A \ni (z,y) \in R$ and $(x,y) \in S^{-1}$

$\Rightarrow (x,y) \in S^{-1}$ and $(y,z) \in R^{-1}$

$\Rightarrow (x,z) \in R^{-1} \circ S^{-1}$ $\quad \therefore (S \circ R)^{-1} \subseteq R^{-1} \circ S^{-1}$ — ①

Now, let $(x,z) \in R^{-1} \circ S^{-1}$

$\Rightarrow \exists y \in A$ s.t $(x,y) \in S^{-1}$ and $(y,z) \in R^{-1}$

$\Rightarrow \exists y \in A$ s.t $(z,y) \in R$ and $(y,x) \in S$

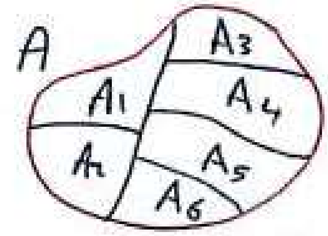
$\Rightarrow (z,x) \in S \circ R$

$\Rightarrow (x,z) \in (S \circ R)^{-1}$

$\therefore (R^{-1} \circ S^{-1}) \subseteq (S \circ R)^{-1}$ — ②

from ① and ② we get $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Def.:- Let A be a set by a تجزئة partition of A we mean a family $\{A_i\}_{i\in I}$ of nonempty subset of A with the following properties.



① $\forall i,j \in I$, $A_i \cap A_j = \emptyset$ or $A_i = A_j$

② $A = \bigcup_{i\in I} A_i$

$\{A_1, A_2, A_3, A_4, A_5, A_6\}$ is a partition of A.

Ex.:- Z_e, Z_o

① $Z_e \cap Z_o = \emptyset$

② $Z_e \cup Z_o = Z \Rightarrow Z_e, Z_o$ is partition of Z.

Ex.:- $A = \{a, b, c, d, e\}$

$A_1 = \{a, b\}$, $A_2 = \{c, d\}$, $A_3 = \{e\}$

$\Rightarrow A_1, A_2, A_3$ is a partition of A.

Ex.:- let $A = \{0, 5, 7, 9\}$

$P_1 = \{\{0, 5, 7\}, \{9\}\}$ is partition of A.

$P_2 = \{\{0, 9\}, \{7\}\}$ is not partition of A.

(لعدم وجود الـ 5)

Ex. :- let $A = \mathbb{Z}$ and ~~and~~ $P = \{\mathbb{Z}_o, \mathbb{Z}_e\}$ Partition of A. (28)

Find A/R (quolient set of A by R).

نوجد صفوف التكافؤ

Sol. :- defined a relation $R = \{(a,b) : a - b = 2k, k \in \mathbb{Z}\}$

1- R is equivalence relation

Q :- Let $R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : a - b = 2k, k \in \mathbb{Z}\}$ Find A/R.

$[0] = \{x \in \mathbb{Z}, \exists k \in \mathbb{Z}, x - 0 = 2k\}$

$= \{-----, -4, -2, 0, 2, 4, ----\} = \mathbb{Z}_e$

$[1] = \{x \in \mathbb{Z}, \exists k \in \mathbb{Z}, x - 1 = 2k\}$

$= \{x \in \mathbb{Z}, \exists k \in \mathbb{Z}, x = 2k + 1\}$

$= \{-----, -3, -1, 1, 3, 5, ----\} = \mathbb{Z}_o$

Note: A/R is the set of all equivalence classes of R in A.

$A/R = \{[0], [1]\} = \{\mathbb{Z}_e, \mathbb{Z}_o\}$.

* Functions * or Mappings

Defintion :- A function (or (mapping)) f from A to B $(f : A \longrightarrow B)$ is a set of ordered pairs such that no two distinct pairs have the some first componedt. Thus

$(x, y_1) \in f$ and $(x, y_2) \in f$ implies $y_1 = y_2$

تعرف الدالة بشكل ازواج مرتبة

$D_f = \{x : (x, y) \in f$ for some $y\}$

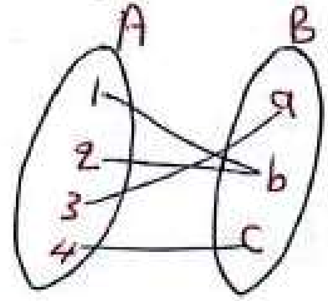
$R_f = \{y : (x, y) \in f$ for some $x\}$

Ex.:- $f = \{(-1,0),(0,0),(1,2),(2,1)\}$ هذه علاقة

$D_f = \{-1,0,1,2\}$, $R_f = \{0,1,2\}$

Ex.:- let $A = \{1,2,3,4\}$, $B = \{a,b,c\}$

$f = \{(1,b),(2,b),(3,a),(4,c)\}$



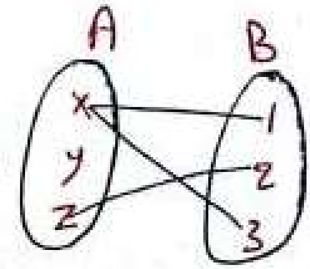
$\therefore f$ is a function

Ex.:- Let $A = \{x,y,z\}$, $B = \{1,2,3\}$

$f = \{(x,1),(x,3),(z,2)\}$ f ليس دالة

لان x لها صورتين ① ③

$\therefore f$ is not function و y ليس لها صورة



Definition:- A function (or mapping) $f: A \to \boxed{B}$ is said to be surjective (onto) if it has the following property:-

$$\forall y \in B, \exists x \in A \text{ s.t } y = f(x)$$

y صورة لـ x

or $\operatorname{Im} f = B$

مجموعة الصور

هو المجال المقابل

Ex.:- $f: Z \longrightarrow N$ with $f(x) = x+3$

شاملة لان المجال المقابل هو المدى = N

f is a function for all $x \in \{x : x \geq -2\}$

(29) $y = x+3$

$y = 0+3 = 3$

$y = 1+3 = 4$

$y = -2+3 = 1 \in N$

Def.:- A function $f: A \to B$ is said to be injective (one-to-one) if it has the following property:-

$\forall x_1, x_2 \in A$ if $f(x_1) = f(x_2)$, then $x_1 = x_2$ دائماً نستخدم للبراهين

or $\forall x_1, x_2 \in A$ if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. هذه للأمثلة

Def.:- A function $f: A \to B$ is said to be bijective متقابلة if it is both surjective and injective.

Example:-

$3 \in N$

$\sqrt{3} \notin N$

1- Let $f: N \longrightarrow N$ such that $f(x) = x^2$, $\forall x \in N$

f is not onto $Im f \neq N$.

$y = x^2$

$\Rightarrow x = \mp\sqrt{y}$

f is one to one, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

f is not bijective function.

2- Let $f: R \to R$ such that $f(x) = 2x^3 + 7$, $\forall x \in R$

f is onto $Im f = R$.

$y = 2x^3 + 7$

$y - 7 = 2x^3$

$x^3 = \frac{y-7}{2}$

$x = \sqrt[3]{\frac{y-7}{2}}$

f is one to one, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$.

$1 \neq -1 \Rightarrow f(1) = 9 \neq f(-1)$

f is bijective function.

3- Let $f: \mathbb{R} \longrightarrow [-1,1]$ such that $f(x) = \cos x, \forall x \in \mathbb{R}$.

f is onto $\operatorname{Im} f = [-1,1]$.

f is not one to one.

f is not bijective function. $\quad -1 \neq 1 \Rightarrow f(-1) = f(1)$

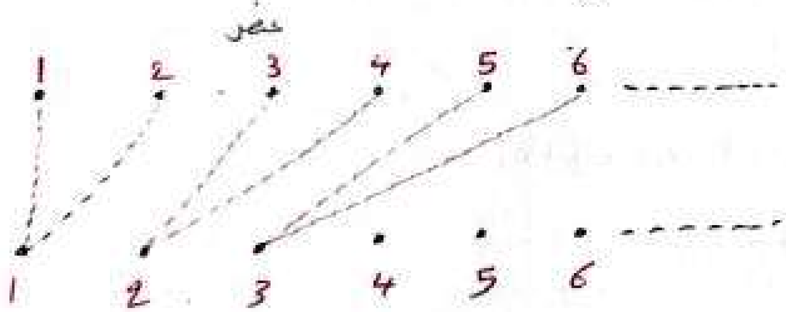
4. Let $f: \mathbb{N} \to \mathbb{N}$, $f(n) = 2n$, $\forall n \in \mathbb{N}$.

f is not onto.

f is one to one.

f is not bijective.

5- $f: \mathbb{N} \to \mathbb{N}$, $f(n) = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ odd} \\ \frac{n}{2} & \text{if } n \text{ even} \end{cases}$



f is onto.

f is not one to one. 3 العنصر $\neq$ العنصر 4 لكن الصور المتساوية الي هي $\underline{2}$

f is not bijective.

Th. Let S and T are sets and $f : S \times T \to T \times S$, then f is bijective function.

Proof:- $f(s,t) = (t,s)$, $\forall\, s,t,\ (s,t) \in S \times T$

first we prove that f is well defined function:

let $(s_1,t_1), (s_2,t_2) \in S \times T$ such that

$$(s_1,t_1) = (s_2,t_2) \Rightarrow s_1 = s_2 ,\ t_1 = t_2$$
$$\Rightarrow (t_1,s_1) = (t_2,s_2)$$
$$\Rightarrow f(s_1,t_1) = f(s_2,t_2)$$

$\therefore f$ is well-defined function.

to prove f is one to one

let $f(s_1,t_1) = f(s_2,t_2)$, $\forall\, (s_1,t_1),(s_2,t_2) \in S \times T$

$\Rightarrow (t_1,s_1) = (t_2,s_2)$.

$\Rightarrow t_1 = t_2 ,\ s_1 = s_2$

$\Rightarrow (s_1,t_1) = (s_2,t_2)$

$\therefore f$ is 1-1.

$\forall\, (t,s) \in T \times S\ \exists\, (s,t) \in S \times T$ s.t $f(s,t) = (t,s)$

$\Rightarrow f$ is onto.

$\Rightarrow f$ is bijective.

هل ممكن ان نكتب الدالة بصورة مثال على ان تكون دالة onto

او 1-1

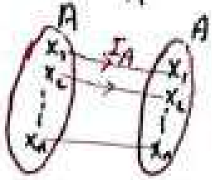
او bijective

اذكر

اكتب

الدالة المحايدة

Ex.:- ① Identity function $I_A : A \to A$, $I_A(x) = x$, $\forall x \in A$.



f is onto.

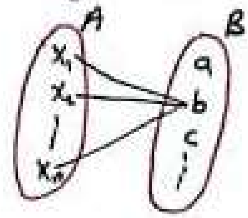
f is one to one.

الدالة الثابتة

② Constant function $K_b : A \to B$, $K_b(x) = b$, $\forall x \in A$.

f is not onto.

f is not one to one.

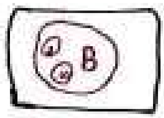


دالة الاحتواء

③ Inclusion function $B \subseteq A$, $E_B : B \to A$, $E_B(x) = x$, $\forall x \in B$.

f is not onto.

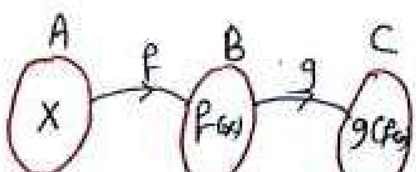
f is one to one. يروح ويرجع ياخذ صورة نفس



Composition of functions تركيب الدوال

Def.:- Let $f : A \to B$ be a function and $g : B \to C$ be a function by the composition of A into C we mean $g \circ f : A \to C$ is a function

$$(g \circ f)(x) = g(f(x)) \ , \ \forall x \in A.$$



Ex.(1):- Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x^2+5$.

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = x$.

$(g \circ f)(x) = g(f(x)) = g(3x^2+5) = 3x^2+5$.

$(f \circ g)(x) = f(g(x)) = f(x) = 3x^2+5$.

Ex.(2):- Let $S = \{x, y, z\}$, $T = \{1, 2, 3\}$, $U = \{a, b, c\}$

$\alpha(x) = 2$, $\alpha(y) = 1$, $\alpha(z) = 3$.

$\beta(1) = b$, $\beta(2) = c$, $\beta(3) = a$.

$(\beta \circ \alpha)(x) = \beta(\alpha(x)) = \beta(2) = c$.

$(\beta \circ \alpha)(y) = \beta(\alpha(y)) = \beta(1) = b$.

$(\beta \circ \alpha)(z) = \beta(\alpha(z)) = \beta(3) = a$.

Ex.(3):- give example show that $f \circ g \neq g \circ f$.

$f: Z \to Z$, $f(x) = x-1$.

$g: Z \to Z$, $g(x) = x^2$.

$(f \circ g)(x) = f(g(x)) = f(x^2) = x^2-1$.

$(g \circ f)(x) = g(f(x)) = g(x-1) = (x-1)^2$.

$\therefore f \circ g \neq g \circ f$.

Ex.(4):- $f: E \to Z$, $f(x) = \frac{x}{2}$.

$g: Z \to N$, $g(x) = x^2$.

find $(f \circ g)(x)$, $(g \circ f)(x)$.

$(f \circ g)(x) = f(g(x)) = f(x^2) = \frac{x^2}{2}$.

$(g \circ f)(x) = g(f(x)) = g\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$

Ex.(5):- Let $f: Z \to E$ given by $f(x) = 2x$.

and $g: E \to Z$ given by $g(u) = \frac{u}{2}$ $\forall u \in E$

$(g \circ f)(x) = g(f(x)) = g(2x) = \frac{2x}{2} = x$.

هذه اسميها الدالة الذاتية

لأن صورة الدالة تساوي العنصر نفسه

$\therefore (g \circ f)(x) = x = I_Z(x)$.

$(f \circ g)(u) = f(g(u)) = f\left(\frac{u}{2}\right) = 2\left(\frac{u}{2}\right) = u$.

$\therefore (f \circ g) = I_E$.

$\forall y \in$ المقابل , $\exists x \in$ المجال , s.t $f(x) = y$

Lemma:- Let $f: A \to B$ is a function, then $g: A \to f(A)$ is bijective function.

صور

الدالة هي عبارة عن ازواج مرتبة

Proof:- Defined $g: A \to f(A)$ by $g(x) = (x, f(x))$

First we must prove that g is well defined

بعكس المتباينة

let $x_1, x_2 \in A$ s.t. $x_1 = x_2$

$\because f$ is function

$\Rightarrow f$ is well-defined $f(x_1) = f(x_2)$

$\Rightarrow (x_1, f(x_1)) = (x_2, f(x_2))$

$\Rightarrow g(x_1) = g(x_2)$ $\therefore g$ is well defined.

Ex.(4):- $f: E \to Z$, $f(x) = \frac{x}{2}$.

$g: Z \to N$, $g(x) = x^2$.

Find $(f \circ g)(x)$, $(g \circ f)(x)$.

$(f \circ g)(x) = f(g(x)) = f(x^2) = \frac{x^2}{2}$.

$(g \circ f)(x) = g(f(x)) = g(\frac{x}{2}) = (\frac{x}{2})^2 = \frac{x^2}{4}$

Ex.(5):- Let $f: Z \to E$ given by $f(x) = 2x$.

and $g: E \to Z$ given by $g(u) = \frac{u}{2}$ $\forall u \in E$

$(g \circ f)(x) = g(f(x)) = g(2x) = \frac{2x}{2} = x$.

هذه اسميها الدالة الذاتية

$\therefore (g \circ f)(x) = x = I_Z(x)$.

لان صورة الدالة تساوي العنصر نفسه

$(f \circ g)(u) = f(g(u)) = f(\frac{u}{2}) = 2(\frac{u}{2}) = u$.

$\therefore (f \circ g) = I_E$.

$\forall y \in$ المقابل, $\exists x \in$ المجال, s.t $f(x)=y$

Lemma:- Let $f: A \to B$ is a function, then $g: A \longrightarrow f(A)$ is bijective function.

صور

الدالة هي عبارة عن ازواج مرتبة

Proof:- Defined $g: A \to f(A)$ by $g(x) = (x, f(x))$

first we must prove that g is well defined

بعكس المتباينة

let $x_1, x_2 \in A$ s.t. $x_1 = x_2$

$\because f$ is function

$\Rightarrow f$ is well-defined $f(x_1) = f(x_2)$

$\Rightarrow (x_1, f(x_1)) = (x_2, f(x_2))$

$\Rightarrow g(x_1) = g(x_2)$ $\therefore g$ is well defined.

To prove g is 1-1 (injective)

let $g(x_1) = g(x_2)$, $\forall x_1, x_2 \in A$

$\Rightarrow (x_1, f(x_1)) = (x_2, f(x_2))$

$\Rightarrow x_1 = x_2 \Rightarrow g$ is 1-1 (one to one).

Now

$\forall (x, f(x)) \in g$, $\exists x \in A$ s.t $g(x) = (x, f(x))$

$\therefore g$ is on to.

$\therefore g$ is bijective.

Theorem:- If $f: A \to B$ and $g: B \to C$ are bijective function
well, onto, 1-1 كلاهما
then $g \circ f: A \to C$ is a bijective function.

Proof:- first we must prove that $g \circ f$ is well defined.

let $x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ (since f is function) f is well

$\Rightarrow g(f(x_1)) = g(f(x_2))$ (since g is function) g is well

$\Rightarrow g \circ f(x_1) = g \circ f(x_2)$

الاصل متساوية او العناصر متساوية

$\because g$ is bijective $\Rightarrow g$ is 1-1 $\Rightarrow f(x_1) = f(x_2)$

$\because f$ is bijective $\Rightarrow f$ is 1-1 $\Rightarrow x_1 = x_2$

$\therefore g \circ f$ is 1-1.

← صورة لـ y

$\because g$ is onto $\Rightarrow \forall z \in C$, $\exists y \in B$ s.t $(y, z) \in g$ or $z = g(y)$.

$\because f$ is onto $\Rightarrow \forall y \in B$, $\exists x \in A$ s.t $(x, y) \in f$ or $y = f(x)$.

← تعرف الدالة بشكل ازواج مرتبة

$\therefore \forall z \in C$, $\exists x \in A$ s.t $(z, x) \in g \circ f$ or $z = (g \circ f)(x)$.

$\therefore g \circ f$ is onto.

$\therefore g \circ f$ is bijective.

Theorem*:- If $f: A \to B$ and $g: B \to C$ are a function and $g \circ f: A \to C$ is bijective then f is 1-1 and g is onto

Proof:-

let $x_1, x_2 \in A$ s.t $f(x_1) = f(x_2)$

$\because g$ is function $\Rightarrow g(y_1) = g(y_2)$, $y_1, y_2 \in B$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$

$\because g \circ f$ is bijective

$\Rightarrow g \circ f$ is 1-1

$\Rightarrow x_1 = x_2$

$\therefore f$ is 1-1

$\because g \circ f$ is onto

$\Rightarrow \forall z \in C, \exists x \in A$ s.t $(x, z) \in g \circ f$ or $z = (g \circ f)(x)$

$\Rightarrow \exists y \in B$ s.t $y = f(x)$ or $(x, y) \in f$.

$\therefore (x, z) \in g \circ f$ and $(x, y) \in f$

$\Rightarrow (y, z) \in g$ $\left((x, y) \in f \text{ and } (y, z) \in g \Rightarrow (x, z) \in g \circ f \right]$

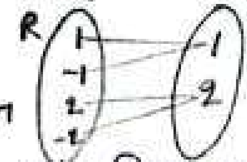
$\Rightarrow z = g(y)$.

$\therefore \forall z \in C, \exists y \in B$ s.t $(g(y)) = z$.

$\therefore g$ is onto

Example:- To prove the converse is not true.

Sol.:- No, since



Let $f : \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = x$

$g : \mathbb{R} \to \mathbb{R}^+$ s.t. $g(x) = x^2$

البرهنة f, g function, $g\circ f$ bijective $\Rightarrow$ f is 1-1, g is onto

العكس $\Uparrow$

f is 1-1 and g is onto

نلاحظ ان

$\to \forall y \in \mathbb{R}^+, \exists x \in \mathbb{R}$ s.t. $(x,y) \in g$ or $y = g(x)$

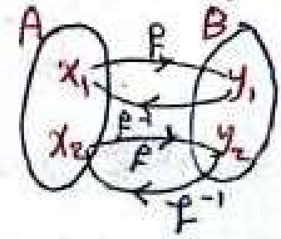
$(g\circ f)(x) = g(f(x)) = g(x) = x^2$

$-1 \neq 1$

$\Rightarrow g(-1) = g(1)$

$\therefore g\circ f$ is not bijective (not one to one, onto)

Def.:- A function $f : A \to B$ is said to be invertible (معكوس) if $f^{-1} : B \to A$ is a function.



Ex.(1):- let $f(x) = x+3$, $f : \mathbb{R} \to \mathbb{R}$

$y = x+3 \Rightarrow x = y-3$

$y = f(x) \Leftrightarrow x = f^{-1}(y)$

$f^{-1}(y) = y-3$ is a function.

كل عنصر في المنطلق له صورة

$f^{-1}(y) = y-3$

اي قيمة نأخذها لـ y الناتج ينتمي لـ R.

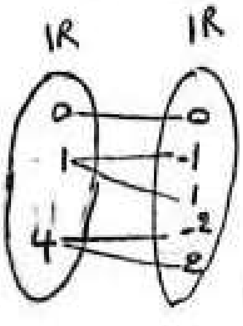
$\therefore$ f is invertible.

Ex.(2):- $g : \mathbb{R} \to \mathbb{R}$ s.t. $g(x) = x^2$

$y = x^2 \Rightarrow x = \mp\sqrt{y}$

لان كل عنصر في المنطلق له صورتين

$\therefore g(x)$ is not invertible



$g = \{(\overset{x}{0}, \overset{g(x)}{0}), (\overset{x}{1}, \overset{g(x)}{1}), (\overset{x}{-1}, \overset{g(x)}{1}), (2,4), (-2,4), \dots\}$

$g^{-1} = \{(0,0), (1,1), (1,-1), (4,2), (4,-2), \dots\}$

$\to g^{-1}(y) = \mp\sqrt{y}$ is not function

Ex. (3):- Let $f: A \to B$, $A = \{1,2,3\}$, $B = \{x,y,z\}$

$f = \{(1,x), (2,y), (3,z)\}$

$f^{-1} = \{(x,1), (y,2), (z,3)\}$

f is invertible.

Theorem:- Let $f: A \to B$ be an invertible function.
Then ① $f^{-1} \circ f = I_A$ ② $f \circ f^{-1} = I_B$

$f^{-1}(y = f(x))$
$f^{-1}(y) = f^{-1} f(x)$
$f^{-1}(y) = x$

Proof:- ① Let $x \in A$ and $y = f(x)$ then $x = f^{-1}(y)$ من تعريف f^{-1}

thus $[f^{-1} \circ f](x) = f^{-1}(f(x)) = f^{-1}(y) = x = I_A$.

② Let $x \in A$ and $y = f(x)$, then $x = f^{-1}(y)$

thus $[f \circ f^{-1}](y) = f(f^{-1}(y)) = f(x) = y = I_B$.

Theorem:- If $f: A \to B$ is bijective function, then f^{-1} is a bijective function.

$f^{-1}: B \to A$

Proof:- let $y_1, y_2 \in B$ s.t. $f^{-1}(y_1) = f^{-1}(y_2)$

$\Rightarrow f(f^{-1}(y_1)) = f(f^{-1}(y_2))$

$\Rightarrow (f \circ f^{-1})(y_1) = (f \circ f^{-1})(y_2)$

$\Rightarrow I_B(y_1) = I_B(y_2)$

$\Rightarrow y_1 = y_2$ $\therefore f^{-1}$ is one to one.

let $y \in B$ $\because f$ is bijective $\Rightarrow f$ is onto.

$\Rightarrow \exists x \in A$ s.t. $y = f(x)$ or $(x,y) \in f$

$\Rightarrow \forall x \in A, \exists y \in B$ s.t. $f^{-1}(y) = x$ or $(y,x) \in f^{-1}$

$\therefore f^{-1}$ is onto $\Rightarrow f^{-1}$ is bijective.

Theorem:- Let $f: A \to B$ and $g: B \to A$ be functions.
If $g \circ f = I_A$ and $f \circ g = I_B$ then $f: A \to B$ is bijective and
$g = f^{-1}$ = invertible(f).

Proof:- let $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$

$$\Rightarrow g(f(x_1)) = g(f(x_2))$$

$$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$$

$$\Rightarrow I_A x_1 = I_A x_2$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$ is one to one.

if $y \in B$ then $\underline{y} = I_B(y) = (f \circ g)(y) = f(\underline{g(y)})$

if y is any element of B, then $\underline{y} = f(\underline{x})$.

where $\boxed{x = g(y)} \xrightarrow[x = f^{-1}(y)]{\text{يلزم نبرهن}}$

$y = f(\underline{g(y)})$

$y = f(\underline{x})$

$\therefore x = g(y)$

$\therefore f$ is onto.

we prove $g = f^{-1} \to$ $\begin{array}{l} x = g(y) \\ x = f^{-1}(y) \end{array}$ and $\begin{array}{l} x = f^{-1}(y) \\ x = g(y) \end{array}$

$$\Rightarrow f(x) = f(g(y)) = (f \circ g)(y) = I_B(y) = y$$

$$\Rightarrow \boxed{x = f^{-1}(y)}$$

$$\Leftarrow x = f^{-1}(y) \Rightarrow y = f(x)$$

$$\Rightarrow g(y) = g(f(x)) = (g \circ f)(x) = I_A(x) = \underline{x}.$$

thus $\forall y \in B$, $x = f^{-1}(y)$ iff $x = g(y)$ that is $f^{-1}(y) = g(y)$.

by Th. [$f: A \to B$, $g: A \to B$ then $f = g$ iff $f(x) = g(x)$, $\forall x \in A$].

$\therefore f^{-1} = g$.

Theorem:- If $f: A \to B$, $g: B \to C$ and $g \circ f: A \to C$ are function then:-

① if f and g are injective, then $g \circ f$ is injective.

② if f and g are onto, then $g \circ f$ is onto.

③ if f and g are bijective, then $g \circ f$ is bijective.

Proof(1):- $\forall x_1, x_2 \in A$, $(g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow f(x_1) = f(x_2)$ (since g is 1-1)

$\Rightarrow x_1 = x_2$ (since f is 1-1)

$\therefore g \circ f$ is injective.

② $\forall z \in C$, then $\exists y \in B$ s.t. $z = g(y)$ or $(y, z) \in g$ (since g is onto)

since $y \in B$ $\exists x \in A$ s.t. $y = f(x)$ or $(x, y) \in f$ (since f is onto).

that is $\forall z \in C$, $\exists x \in A$ s.t. $z = g(f(x)) = (g \circ f)(x) \Rightarrow$

$g \circ f$ is onto.

③ From ① and ② we get

$g \circ f$ is bijective.

$(x, y) \in f$

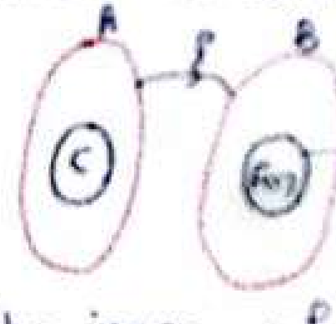
$(y, z) \in g$

$\Rightarrow (x, z) \in g \circ f$

Direct Images and Inverse Images under Function

الصورة المباشرة و معكوس الصورة المباشرة تحت تأثير الدالة

Def.:- Let $f: A \to B$ be a function, if C is any subset of A, the direct image of C under f, which we write $f(C)$ is the following subset of B.



the direct image

الصورة المباشرة

$$f(C) = \{y \in B : \underline{\exists x \in C} \text{ s.t } y = f(x)\}$$

That is, $f(C)$ is the set of all the image of elements in C. هي المجموعة كل صور العناصر الموجودة في المجموعة C

Ex.:- $f: \mathbb{Z} \longrightarrow \mathbb{R}$, $f(x) = x^2 + 2$

$C = \{-3, -1, 0, 1, 3\} \longrightarrow$ مجموعة جزئية من Z

$f(C) = \{2, 3, 11\} = \{f(-3), f(-1), f(0), f(1), f(3)\}$ صور هذه العناصر

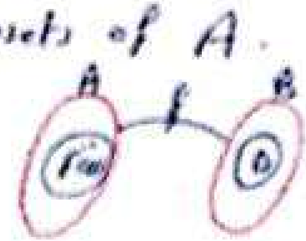
مجموعة جزئية من R

Def.:- The inverse image

Let $f: A \to B$ be a function, if D is any subsets of B, the inverse image of D under f, which we write $f^{-1}(D)$, is the following subsets of A.

صورها في D

$$f^{-1}(D) = \{x \in A : f(x) \in D\}$$



That is, $f^{-1}(D)$ is the sets of all the pre-image of element in D.

مجموعة كل معكوس الصور الموجودة في D

يعني D صور العناصر من أصل الانفرس

علينا بنجيب معكوس الصور

Note:- If $\{a\}$ and $\{b\}$ are singletons, we will write $F^{-1}(a)$ for $F^{-1}(\{a\})$, $F^{-1}(b)$ for $F^{-1}(\{b\})$ اذا المجموعة تتكون من عنصر واحد شول نكتب الاقواس بها

Ex.: ① Let $f: A \to B$, $A = \{a, b, c, d\}$
$B = \{1, 2, 3, 4\}$

$f(a) = 1$, $f(b) = 3$, $f(c) = 2$, $f(d) = 4$.

الصورة المباشرة لـ A $F(A) = \{\overset{f(a)}{1}, \overset{f(b)}{3}, \overset{f(c)}{2}, \overset{f(d)}{4}\}$

معكوس الصورة $F^{-1}(B) = \{a, b, c, d\}$

② let $f: \mathbb{Z} \to \mathbb{R}$, $f(x) = x^2 + 3$

$A = \{1, 0, 5, -6, 2\} \subset \mathbb{Z}$

$B = \{3, 4, 25, 19, 9, 2\} \subset \mathbb{R}$.

$F(A) = \{f(1), f(0), f(5), f(-6), f(2)\}$

$= \{\overset{1+3}{4}, \overset{0+3}{3}, 28, 39, 7\}$

$F^{-1}(B) = \{x \in \mathbb{Z} : x^2 + 3 \in B\}$ الصورة العكسية يعني اي عنصر صورة 3 بعمل هذه الدالة

$= \{x^2+3=3,\ x^2+3=4,\ x^2+3=25,\ x^2+3=19,\ x^2+3=9,\ x^2+3=2\}$

$F^{-1}(B) = \{0, 1, -1, -4, 4\}$

$x^2 + 3 = 3 \Rightarrow x^2 = 0 \Rightarrow x = 0 \in \mathbb{Z}$.

$x^2 + 3 = 25 \Rightarrow x^2 = 22 \Rightarrow x = \mp\sqrt{22} \notin \mathbb{Z}$

$x^2 + 3 = 4 \Rightarrow x^2 = 1 \Rightarrow x = \mp 1 \in \mathbb{Z}$

$x^2 + 3 = 19 \Rightarrow x^2 = 16 \Rightarrow x = \mp 4 \in \mathbb{Z}$

$x^2 + 3 = 9 \Rightarrow x^2 = 6$
$\Rightarrow x = \mp\sqrt{6} \notin \mathbb{Z}$

$x^2 + 3 = 2 \Rightarrow x^2 = -1$
$\Rightarrow x = \mp\sqrt{-1} \notin \mathbb{Z}$

3- Let $f : \mathbb{R}-\{0\} \longrightarrow \mathbb{R}$ be a function

$f(x) = \frac{1}{x}$, $\forall\, x \in \mathbb{R}-\{0\}$, $C = (0,1]$

مجموعة جزئية

$F(C) = \{y \in \mathbb{R} : 1 \leq y < \infty\}$

Theorem:- Let $f : X \to Y$ be a function and $A \subseteq X$, $B \subseteq X$, if $A = B$ then $F(A) = f(B)$.

Proof:- suppose that $A = B$ and let $y \in F(A)$

$\Rightarrow \exists\, x \in A$ s.t. $y = f(x)$ $\quad F(C) = \{y \in B, \exists x \in C \text{ s.t. } y = f(x)\}$

$\because A = B \Rightarrow x \in B$ s.t. $y = f(x)$

$\Rightarrow f(x) \in F(B)$ s.t. $y = f(x)$

$\Rightarrow y \in F(B) \Rightarrow F(A) \subseteq F(B)$ --- ①

let $z \in F(B)$

$\Rightarrow \exists\, e \in B$ s.t $z = f(e)$

$\because A = B \Rightarrow \exists\, e \in A$ s.t. $z = f(e)$

$\Rightarrow f(e) \in F(A)$ s.t. $z = f(e) \Rightarrow z \in F(A)$.

$\therefore F(B) \subseteq F(A)$ --- ②

From ① and ② we get $F(A) = F(B)$.

Ex.:- to show that the converse of this theorem is not true.

Sol.:- No, since

let $f: \mathbb{Z} \to \mathbb{R}$, $f(x) = x^2$

$A = \{2, 1, 3\}$, $B = \{2, 1, -3\}$

$F(A) = \{4, 1, 9\}$. $F(B) = \{4, 1, 9\}$

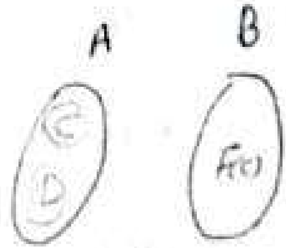
$F(A) = F(B)$ but $A \neq B$.

Theorem:- Let $f: A \to B$ be a function and if $C \subseteq A$, $D \subseteq A$, then

① $F(C \cup D) = F(C) \cup F(D)$.

② $F(C \cap D) \subset F(C) \cap F(D)$.

③ $F(C) - F(D) \subset F(C - D)$.



Proof(1):- let $y \in F(C \cup D)$, $y \in B$, since f is a function

$\Rightarrow \forall y \in F(C \cup D), \exists x \in (C \cup D)$ s.t $y = f(x)$

$\Rightarrow \forall y \in F(C \cup D), \exists x \in C$ or $x \in D$ s.t $y = f(x)$

$\Rightarrow f(x) \in F(C)$ or $f(x) \in F(D)$

$\Rightarrow y \in F(C)$ or $y \in F(D)$

$\Rightarrow y \in F(C) \cup F(D)$

$\therefore F(C \cup D) \subseteq F(C) \cup F(D)$ ---①

$\Leftarrow$ H.w

Proof(2):- let $y \in F(C \cap D)$, $y \in B$

$\because$ f is a function

$\Rightarrow \forall y \in F(C \cap D), \exists x \in (C \cap D)$ s.t $y = f(x)$

$\Rightarrow \forall y \in F(C \cap D), \exists x \in C$ and $x \in D$, $y = f(x)$

صورة x تنتمي F(c)

$\Rightarrow f(x) \in F(C)$ and $f(x) \in F(D)$

$\Rightarrow y \in F(C)$ and $y \in F(D)$

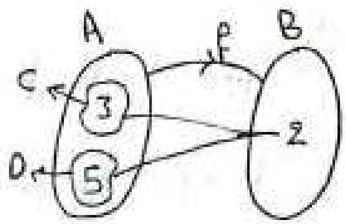
$\Rightarrow y \in F(C) \cap F(D)$.

$\therefore F(C \cap D) \subset F(C) \cap F(D)$.

* To show that $F(C) \cap F(D) \not\subset F(C \cap D)$.

let $f: A \to B$, $A = \{3, 5\}$, $B = \{2\}$

and $f(x) = 2$, $\forall x \in A$



suppose that $C = \{3\}$, $D = \{5\}$

$C \cap D = \emptyset$

$F(C \cap D) = F(\emptyset) = \emptyset$

$F(C) = \{2\}$, $F(D) = \{2\}$

$\Rightarrow F(C) \cap F(D) = \{2\}$

$\Rightarrow \{2\} \not\subset \emptyset$

$\therefore F(C) \cap F(D) \not\subset F(C \cap D)$.

Proof (3):- let $y \in F(C) - F(D)$, $y \in B$

$\Rightarrow y \in F(C)$ and $y \notin F(D)$

$\because f$ is a function

$\Rightarrow \exists\, x \in A$ s.t. $y = f(x)$

$\Rightarrow \exists\, x \in C$ and $x \notin D$ s.t. $y = f(x)$

$\Rightarrow \exists\, x \in (C - D)$, s.t $y = f(x)$

$\Rightarrow f(x) \in F(C - D)$, s.t. $y = f(x)$

$\Rightarrow y \in F(C - D)$.

$\therefore F(C) - F(D) \subset F(C - D)$.

* To show that $F(C - D) \not\subset F(C) - F(D)$.

نفس المثال السابق Sol:- let $f : A \to B$, $A = \{3, 5\}$, $B = \{2\}$,

and $f(x) = 2$, $\forall\, x \in A$

suppose that $C = \{3\}$, $D = \{5\}$

$C - D = \{3\}$

$F(C - D) = F(3) = \{2\}$

$F(C) - F(D) = \{2\} - \{2\} = \varnothing$

$\therefore F(C - D) = \{2\} \not\subset \varnothing = F(C) - F(D)$.

$\therefore F(C - D) \not\subset F(C) - F(D)$.

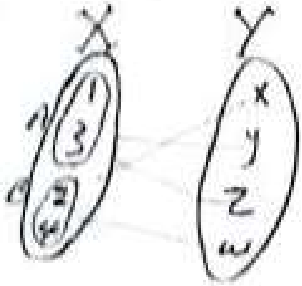
Ex: $F(A\cap B) \not\subset F(A)\cap F(B)$

$F(A-B) \not\subset F(A)-F(B)$.

Sol: $X=\{1,2,3,4\}$, $Y=\{x,y,z,w\}$

$A=\{1,3\}$, $B=\{2,4\}$

$F(A)=\{z,y\}$, $F(B)=\{x,z\}$



$F(A)\cap F(B)=\{z\}$

$A\cap B=\emptyset$, $F(A\cap B)=\emptyset$

$\therefore F(A)\cap F(B) \not\subset F(A\cap B)$.

$A-B=\{1,3\} \Rightarrow F(A-B)=\{z,y\}$

$F(A)-F(B)=\{y\}$ $\therefore F(A-B) \not\subset F(A)-F(B)$.

Ex: $f: R \to R$, $f(x)=x^2+1$, $\forall x \in R$ and

$D=\{17\}$ or $D=\{5,10\}$, find $f^{-1}(D)$.

Sol: $F^{-1}(D)=F^{-1}(\{17\})=\{x\in R : f(x)\in D\}$

$=\{x\in R : x^2+1=17\}$

$=\{x\in R : x^2=16\}=\{-4,4\}$.

$F^{-1}(D)=F^{-1}(\{5,10\})=\{x\in R : f(x)\in D\}$

$=\{x\in R : x^2+1=5 , x^2+1=10\}$

$=\{x\in R : x^2=4 , x^2=9\}$

$=\{-2,2,-3,3\}$.

Theorem:- Let $f: X \to Y$ be a function and $C \subset Y$, $D \subset Y$, if $C = D$, then $f^{-1}(C) = f^{-1}(D)$.

Proof:- let $x \in f^{-1}(C)$, $x \in X$

$\because$ f is a function

$\Rightarrow f(x) \in C$

$\because C = D \Rightarrow f(x) \in D$

$\Rightarrow x \in f^{-1}(D)$ $\quad \therefore f^{-1}(C) \subset f^{-1}(D)$ ---- ①

Now, let $y \in f^{-1}(D)$, $y \in Y$

$\because$ f is a function

$\Rightarrow f(y) \in D$

$\because C = D \Rightarrow f(y) \in C$

$\Rightarrow y \in f^{-1}(C)$ $\quad \therefore f^{-1}(D) \subset f^{-1}(C)$ ---- ②

From ① and ② we get

$f^{-1}(C) = f^{-1}(D)$.

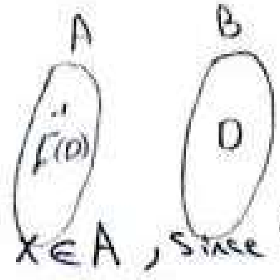
Theorem:- Let $f: A \to B$ be a function and $C \subseteq B$, $D \subseteq B$, then

$F^{-1}(D) = \{x \in A : f(x) \in D\}$

① $F^{-1}(C \cup D) = F^{-1}(C) \cup F^{-1}(D)$. H.w

② $F^{-1}(C \cap D) = F^{-1}(C) \cap F^{-1}(D)$.

③ $F^{-1}(C - D) = F^{-1}(C) - F^{-1}(D)$.



Proof (2):- let $x \in F^{-1}(C \cap D)$, $x \in A$, since f is fun.

$\Rightarrow f(x) \in C \cap D$

$\Rightarrow f(x) \in C$ and $f(x) \in D$

$\Rightarrow x \in F^{-1}(C)$ and $x \in F^{-1}(D)$

$\therefore F^{-1}(C \cap D) \subset F^{-1}(C) \cap F^{-1}(D)$ --- ①

by the same way $F^{-1}(C) \cap F^{-1}(D) \subset F^{-1}(C \cap D)$ --- ②

From ① and ② we get

$F^{-1}(C \cap D) = F^{-1}(C) \cap F^{-1}(D)$.

Proof(3):- let $x \in f^{-1}(C-D)$, $x \in A$, f is function

$\Rightarrow f(x) \in (C-D)$

$\Rightarrow f(x) \in C$ and $f(x) \notin D$

$\Rightarrow x \in f^{-1}(C)$ and $x \notin f^{-1}(D)$

$\therefore x \in [f^{-1}(C) - f^{-1}(D)]$

$\Rightarrow f^{-1}(C-D) \subset f^{-1}(C) - f^{-1}(D)$.

4) let $B=\{1,2,3,4\}$ and $C=\{a,b,c\}$. List fore different surjective functions from B to C.

$f_1 = \{(1,a),(2,a),(3,b),(4,c)\}$

$f_2 = \{(1,b),(2,b),(3,a),(4,c)\}$

$f_3 = \{(1,c),(2,a),(3,b),(4,a)\}$

$f_4 = \{(1,a),(2,b)(3,c),(4,b)\}$