Introduction

functional analysis is one of many branches of mathematics. which has application in Calculus, differential equations, numerical analysis, topology and algebra. A first step on functional analysis start with definition of vector spaces.

vector space is the famous K-module known in module theory. when the module is ~~a~~ a commutative group defined on a ring.

In this ~~lectures~~ lecuters. The symbol $\bar{X}$ will always used to denoted to vector space, the elements of $\bar{X}$ are called vectors

K will denoted to field of either real number $\mathbb{R}$ or complex number $\mathbb{C}$. The elements of K are called scalars.

1.1 Vector space

1.1.1 Definition of vector space

A vector space (or linear space) over a field K is a nonempty set ~~X~~ X with two operations vector addition and multiplication of vectors by scalars and satisfying the following condition, The vector space al usually denoted by $(X, +, \cdot, K)$

A vector addition. $(X, +)$

The set X with operation $+$ it will be $(X, +)$ is commutative group.

(A1) $+ : X \times X \longrightarrow X$ is a binary operation closed defined on X such that

$x+y \in X$ for all $x, y \in X$

(A2) Associative law

$x+(y+z) = (x+y)+z$ for all $x, y, z \in X$

(A3) Identity element

There exists an element $0 \in X$ such that $x + 0 = 0 + x = x$ for all $x \in X$

(A4) Inverse element

For all $x \in X$, there exist $-x \in X$ called inverse element of x for addition operation such that $x + (-x) = (-x) + x = 0$

(A5) commutative operation

~~for all x, y ∈ X~~

$x + y = y + x$ for all $x, y \in X$

S Multiplication by Scalar $(X, \cdot, K)$

There is a mapping from $K \times X$ into X satisfies

(S1) closed under scalar multiplication

$\cdot : K \times X \longrightarrow X$ is a mapping such that $\alpha \cdot x \in X$ for all $x \in X$, $\alpha \in K$

note that the mapping $\cdot : X \times K \longrightarrow X$ is represent $x \cdot \alpha \in X$. since K is always commutative we have $\alpha \cdot x = x \cdot \alpha$.

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(S2) Scalar multiplication distrubtive into vector addition

$$\alpha \cdot (X+y) = \alpha \cdot X + \alpha \cdot y$$ توزيعية

(S3) Scalar multiplication distrubtive to scalar addition

$$(\alpha + B) \cdot X = \alpha \cdot X + \beta \cdot X$$

(S4) associative law

$$(\alpha \cdot B) \cdot X = \alpha \cdot (B \cdot X)$$

(S5) Scalar Identity

$$1 \cdot X = \bar{X}$$

where $1 \in \bar{K}$ is the identity elements and $x, y \in X$, $\alpha, B \in K$

The vector space is called real vector space when $K = \mathbb{R}$ and called complex vector space when $K = \mathbb{C}$.

<u>Note</u> The are two type of zero in vector space.

(*) The zero vector which is the identity element for the vector addition.

(**) The zero scalar which is the identity element for the scalar addition of the field k.

The reader must be ~~regocnize~~ recognize the difference between them for the contex.

1.1.2 Example

~~Let~~ we have the set R^n where R is real number. prove that $(R^n, +, \cdot, R)$ is a vector space when $n \in N$ ~~wi~~ N denoted to set of natural ~~num~~ number.

and $+$ is denoted to usual addition on R^n and $\cdot$ is usual multiplication

$$R^n = \underbrace{R \times R \times \cdots R}_{n\text{-times}}$$

is the cartision product.

~~Solution~~

Let $x \in R^n$, $n \in N$

$x = (x_1, x_2, \dots, x_n)$ where $x_i \in R$ $1 \leq i \leq n$

or we can write $x = (x_i)_{i=1}^{n}$ $x_i \in R$

The vector addition

$+ : R^n \times R^n \longrightarrow R^n$

$x = (x_1, x_2, \cdots, x_n)$, $y = (y_1, y_2, \cdots y_n)$

$x+y = (x_1+y_1, x_2+y_2, \ldots\ldots x_n+y_n)$

or $x = (x_i)_{i=1}^{n}$, $y = (y_i)_{i=1}^{n}$

$x+y = (x_i+y_i)_{i=1}^{n}$ $\quad x_i, y_i \in R$

$\cdot : R \times R^n \longrightarrow R$

$\alpha \cdot x = (\alpha x_1, \alpha x_2, \cdots \alpha x_n)$

$= (\alpha x_i)_{i=1}^{n}$ $\quad x_i \in R$, $\alpha \in R$

Scalar multiplication

<u>Solution</u>

At first we to show that $(R^n, +)$ is a commutative group

Let $x, y \in R^n$, then

(A1) $x = (x_1, \cdots x_n)$, $y = (y_1, \cdots y_n)$

$x+y = (x_1+y_1, \cdots x_n+y_n)$

$\because x_i, y_i \in R \longrightarrow x_i+y_i \in R$ for all $1 \le i \le n$

$\therefore (x_1+y_1, \cdots\cdots x_n+y_n) \in R^n$

So $x+y \in R^n$ is closed under addition. that is for all $x, y \in R^n$

A2 Let $x, y, z \in R^n$

$x = (x_1, \dots x_n)$, $y = (y_1, \dots y_n)$, $z = (z_1, \dots z_n)$
$= (x_i)_{i=1}^n$ $\quad = (y_i)_{i=1}^n$ $\quad z = (z_i)_{i=1}^n$

$$(x+y)+z = \left((x_i)_{i=1}^n + (y_i)_{i=1}^n\right) + (z_i)_{i=1}^n$$
$$= (x_i+y_i)_{i=1}^n$$
$$= \left((x_i+y_i)+z_i\right)_{i=1}^n$$
$$= \left(x_i+(y_i+z_i)\right)_{i=1}^n$$
$$= (x_i)_{i=1}^n + (y_i+z_i)_{i=1}^n$$
$$= x + (y+z)$$

that is for all $x, y, z \in R^n$

$\therefore R^n$ is associative

A3 There is exist $0 = (0, 0, \dots 0) \in R^n$ such that

$$0+x = (0, \dots 0) + (x_1, \dots x_n)$$
$$= (0+x_1, \dots 0+x_n)$$
$$= (x_1, \dots x_n)$$
$$= x$$

zero scalar

different

~~Similarly~~ Similarly $x+0 = 0$

zero vector

~~For~~ That is hold for all $x \in R^n$

(A4) Invers element

Let $x \in R^n$, $x = (x_1, \cdots x_n)$

$-x = (-x_1, \cdots -x_n)$

So $x + {}^-x = (x_1, \text{———} x_n) + (-x_1, \quad , -x_n)$

$= (x_1 + {}^-x_1, x_2 + {}^-x_2, \cdots, x_n + {}^- x_n)$

$= (0, \ , 0, \ \cdots \ , 0)$

$= 0$

Similarly we get $-x + x = 0$

that is hold forall $x \in R^n$

(A5) commutative operation

$x = (x_1, \text{—} x_n)$, $y = (y_1, \text{——} y_n)$

$x + y = (x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n)$

$= (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$

$= (y_1 + x_1, y_2 + x_2, \cdots, y_n + x_n)$

$= (y_1, y_2, \cdots y_n) + (x_1, x_2, - x_n)$

$= y + x$

That is hold forall $x, y \in R^n$

so ~~$(R^n, +)$~~ R^n is commutative

so $(R^n, +)$ is commutative group.