Introduction

from our study; we recall the dot product between to vector, ~~in~~ Let $x, y \in \mathbb{R}^3$

$x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$

The dot product is

$x \cdot y = x_1 \cdot y_1 + x_2 y_2 + x_3 y_3$

if $y = x$ we have

$x \cdot x = x_1 \cdot x_1 + x_2 x_2 + x_3 x_3$

$$x \cdot x = \left[(x_1^2 + x_2^2 + x_3^2)^{\frac{1}{2}} \right]^2$$

$$x \cdot x = \| x \|^2$$

we say that x is perpendicular to y if

$x \cdot y = 0$

the dot product has many important applications.

Hence the question arises whether the dot product and orthogonality can be generalized to arbitrary vector spaces. In fact, this can be done and leads to inner product spaces (pre-Hilbert space) and complete inner product space (Hilbert space).

1-1) # Inner product space

Definition Inner product

Let X be a vector space over the field K. The inner product is a mapping of $X \times X$ into the scalar field K, which always denoted by $\langle \cdot / \cdot \rangle$ or $\langle \cdot , \cdot \rangle$ or $(\, , \,)$

$$\langle \cdot / \cdot \rangle : X \times X \longrightarrow K$$
$$(x, y) \longrightarrow \langle x / y \rangle$$

$\langle x / y \rangle$ is called the inner product of x and y and satisfy the following conditions

IP_1) (Linearity in the first variable)

for all $x_1, x_2, x, y \in X$, $\alpha \in K$

$$\langle x_1 + x_2 / y \rangle = \langle x_1 / y \rangle + \langle x_2 / y \rangle$$
$$\langle \alpha x / y \rangle = \alpha \langle x / y \rangle$$

IP_2) (Positive definition) for all $x \in X$

$$\langle x / x \rangle \geq 0$$

$\langle x / x \rangle = 0$ iff $x = 0$

IP_3) (conjugate symmetry)

$$\langle x / y \rangle = \overline{\langle y / x \rangle}$$

where $\overline{\quad}$ denote complex conjugation

$z = x + yi$, $\bar{z} = x - yi$

Definition Pre Hilbert space or Inner product space

A pre Hilbert space or inner product space is a vector space X with an inner product defined on X

Definition Hilbert space

A Hilbert space is a complete inner product space (pre Hilbert space).

Note (1) ~~Compl~~

complete space means every Cauchy sequence in the space are convergent to point in the same space.

Note (2)

~~From the~~ for the vector space X over the field K $X \times X$ is also vector space over the field K with usual addition and scalar multiplication, so the inner product $\langle \cdot / \cdot \rangle : X \times X \longrightarrow K$ is an functional.

Example ① Let C^2 be a vector space over C then the inner product defined on C^2 is

for $x, y \in C^2$, $x = (x_1, x_2)$
$y = (y_1, y_2)$

$$\langle x, y \rangle = x_1 \cdot \overline{y_1} + x_2 \cdot \overline{y_2}$$

show that ~~...~~ C^2 with $\langle \cdot, \cdot \rangle$ is inner product space

$IP_1)$ Let $x, y, z \in \mathbb{C}^2$ $z = (z_1, z_2)$

$x + y = (x_1 + y_1, x_2 + y_2)$

$$\langle x + y, z \rangle = (x_1 + y_1)\overline{z_1} + (x_2 + y_2)\overline{z_2}$$
$$= x_1\overline{z_1} + x_2\overline{z_2} + y_1\overline{z_1} + y_2\overline{z_2}$$
$$= \langle x, z \rangle + \langle y, z \rangle$$

$\alpha x = \alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$

$$\langle \alpha x, z \rangle = \alpha x_1 \overline{z_1} + \alpha x_2 \overline{z_2}$$
$$= \alpha(x_1\overline{z_1} + x_2\overline{z_2})$$
$$= \alpha \langle x, z \rangle$$

$IP_2)$ $\langle x, x \rangle = x_1\overline{x_1} + x_2\overline{x_2}$

$\because x_1\overline{x_1} = |x_1|^2$

also $x_2\overline{x_2} = |x_2|^2$

where $x_1 = a_1 + b_1 i$
$\overline{x_1} = a_1 - b_1 i$

$$x_1\overline{x_1} = a_1^2 + b_1^2 = \left(\sqrt{a_1^2 + b_1^2}\right)^2 = |x_1|^2 \geq 0$$

similarly $x_2 \cdot \overline{x_2} \geq 0$

so

$\therefore \langle x, x \rangle \geq 0$

Let $\langle x, x\rangle = 0 \iff x_1\overline{x_1} + x_2\overline{x_2} = 0$

$\iff |x_1|^2 + |x_2|^2 = 0$

$\iff |x_1| = 0 \ , \ |x_2| = 0$

$x_1 = 0, \ x_2 = 0$

$x = (x_1, x_2) = (0, 0)$

$IP_3)$ $\langle x, y\rangle = x_1\overline{y_1} + x_2\overline{y_2}$

$$\overline{\langle y, x\rangle} = \overline{y_1\overline{x_1} + y_2\overline{x_2}}$$

$$= \overline{y_1\overline{x_1}} + \overline{y_2\overline{x_2}}$$

$$= \overline{\overline{x_1}}\,\overline{y_1} + \overline{\overline{x_2}}\,\overline{y_2}$$

$$= x_1\overline{y_1} + x_2\overline{y_2}$$

$$= \langle x, y\rangle$$

$\therefore \langle x, y\rangle = \overline{\langle y, x\rangle}$

It is clear that $\langle \cdot, \cdot \rangle$ is a functional which domain is $\mathbb{C}^2$ vectorspace and codomain is base ~~fited~~ field $\mathbb{C}$.

~~Remark:~~
proposition

~~every inner product on vector space X over field K is a norm on X given by $\|x\| = \sqrt{\langle x, x\rangle}$ or $\|x\|^2 = \langle x, x\rangle$~~

If we take $x=(1+i,\ i)\in \mathbb{C}^2$

$y=(2-i,\ 3)$

$z=(1-i,\ 1+i)$

in Example (1)

Then

$$\langle x,y\rangle = x_1\overline{y_1}+x_2\overline{y_2}$$

$$=(1+i)\overline{(2-i)}+i\times\overline{3}$$

$$=(1+i)(2+i)+i\times 3$$

$$=[(2-1)+(2+1)i]+3i$$

$$=1+3i+3i$$

$$=1+6i$$

$$\langle z,y\rangle = z_1\overline{y_1}+z_2\overline{y_2}$$

$$=(1-i)\overline{(2-i)}+(1+i)\overline{3}$$

$$=(1-i)(2+i)+(1+i)3$$

$$=(2+1)+(-2+1)i+3+3i$$

$$=3-i+3+3i$$

$$=6+2i$$

$$\langle x,y\rangle+\langle z,y\rangle = 1+6i+6+2i$$

$$=7+8i$$

now we want to find $\langle x+z,y\rangle$

$\cancel{x+z=}$

$\langle x+z, y\rangle$

$$x+z = (1+i, i)+(1-i, 1+i) = (\underline{2}, 1+2i) \quad x_1+z_1 \quad x_2+z_2$$

$$y = (2-i, 3)$$

$$\langle x+z, y\rangle = x_1+z_1\,\overline{y_1} + (x_2+z_2)\,\overline{y_2}$$

$$= 2\cdot\overline{(2-i)} + (1+2i)\,\overline{3}$$

$$= 2(2+i) + (3+6)i$$

$$= 4+2i + 3+6i$$

$$= 7+8i$$

$\therefore \langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$

Let $\alpha = 3-i$ find $\langle \alpha x, y\rangle$

$$\cancel{\langle \alpha x, y\rangle} = \alpha x = (3-i)(1+i, i)$$

$$= ((3-i)(1+i), (3-i)i)$$

$$= ((3+1)+(-1+3)i, 3i+1)$$

$$= (4+2i, 1+3i)$$

$$(\alpha x, y) = \alpha x_1\overline{y_1} + \alpha x_2\overline{y_2}$$

$$= (4+2i)\overline{(2-i)} + (1+3i)\overline{3}$$

$$= (4+2i)(2+i) + (1+3i)3$$

$$= (8-2) + (4+4)i + 3+9i$$

$$= 6+8i+3+9i$$

$$= 9+17i$$

$$\alpha\cdot\langle x, y\rangle = (3-i)\cdot(1+6i)$$

$$= (3 - -6) + (-1+18)i$$

$$= 9+17i$$

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$\langle x, x \rangle \qquad x = (1+i, i)$

$\langle x, x \rangle = x_1 \overline{x_1} + x_2 \overline{x_2}$

$= (1+i)\overline{(1+i)} + i\,\overline{i}$

$= (1+i)(1-i) + i(-i)$

$= (1^2 + i^2) + 1$

$= 3 \geq 0$

$y = (2-i, 3)$

$\langle y, y \rangle = y_1 \overline{y_1} + y_2 \overline{y_2}$

$= (2-i)\overline{(2+i)} + 3\,\overline{3}$

$= (2-i)(2+i) + 3 \times 3$ 5

$= (2^2 + i^2) + 9$

$= 14$

$z = (1-i, 1+i)$

$\langle z, z \rangle = z_1 \overline{z_1} + z_2 \overline{z_2}$

$= (1-i)\overline{(1+i)} + (1+i)\overline{(1+i)}$

$= (1-i)(1+i) + (1+i)(1-i)$

$= (1^2 + i^2) + (1^2 + i^2)$

$= 2 + 2$

$= 4$

Note: Let $w = a + bi \in \mathbb{C}$, $a, b \in \mathbb{R}$

$\overline{w} = a - bi$

then $w \cdot \overline{w} = (a+bi)(a-bi)$

$= (a^2 - -b^2)\underbrace{(ab - ab)}_{=0}i$

$= a^2 + b^2$

we see that $w \cdot \overline{w} = a^2 + b^2$

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$x = (1+i, i)$

we take $y = (2-i, 3)$

$\langle x, y\rangle = 1+6i$

$\overline{\langle x, y\rangle} = 1-6i$

so $\overline{\langle x, y\rangle} = \langle y, x\rangle$

now we find $\langle y, x\rangle$

$$\begin{aligned}
\langle y, x\rangle &= y_1\overline{x_1} + y_2\overline{x_2} \\
&= (2-i)\overline{(1+i)} + 3\,\overline{i} \\
&= (2-i)(1-i) + 3(-i) \\
&= (2-1) + (-2-1)i - 3i \\
&= 1 - 3i - 3i \\
&= 1 - 6i
\end{aligned}$$

so $\langle y, x\rangle = \overline{\langle x, y\rangle}$

~~let $x_n = \left(2+\frac{1}{n} + \left(1-\frac{1}{n}\right)i, 3\right)$ is $(x_n)_{n=1}^{\infty}$ sequence~~

~~$\mathbb{C}^2$ ~~find the limit point of it~~ and prove that $(x_n)_{n=1}^{\infty}$ is convergent~~

~~we see that $x_n \to x_0$ when $x_0 = (2+i, 3)$~~

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