Advanced Calculus (2)

Triple integrals التكاملات الثلاثية

Definition: Let D be the region defined by the inequalities $a \le x \le b$ $h_1(x) \le y \le h_2(x)$, $g_1(x,y) \le z \le g_2(x,y)$. If $f(x,y,z)$ is continuous function of three variables, then the triple integral is

$$V = \iiint_D f(x,y,z)dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x,y)}^{g_2(x,y)} dz\, dydx$$

Definition: The volume of a closed, bounded region D in space is

$$V = \iiint_D dV$$

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Example 1: Find the value of triple integral for the function $f(x,y,z) = 2xyz$ and bounded by surface $z = 2 - \frac{1}{2}x^2$ and the planes $z = 0$, $y = x$, $y = 0$.

Solution:

$$V = \iiint_D 2xyz\, dV = \int_0^2 \int_0^X \int_0^{2-\frac{1}{2}x^2} 2xyz\ \ dz\, dydx$$

$$= \int_0^2 \int_0^x xyz^2\Big]_0^{2-\frac{1}{2}x^2} \text{dy dx}$$

$$= \int_0^2 \int_0^x xy(2 - \frac{1}{2}x^2)^2 \text{dy dx} = \int_0^2 \int_0^x xy\left[4 - 2x^2 + \frac{1}{4}x^4\right] \text{dydx} =$$

$$\int_0^2 \int_0^x \left[4xy - 2x^3y + \frac{1}{4}x^5y\right] \text{dydx} = \int_0^2 (2xy^2 - x^3y^2 + \frac{1}{8}x^5y^2)\Big]_0^x dx$$

$$= \int_0^2 \left[2x^3 - x^5 + \frac{1}{8}x^7\right] dx = \frac{x^4}{2} - \frac{x^6}{6} + \frac{1}{64}x^8\Big]_0^2 = 8 - \frac{64}{6} + \frac{256}{64} = \frac{4}{3}$$

Example 2: Use **triple integral** to find the volume in the first octant (الثمن) of the region that bounded by two surfaces $x^2 + 4y^2 = 4z$ and $x^2 + 4y^2 = 48 - 8z$.

Solution:

$$V = \iiint_D dV = \int_0^2 \int_0^{2\sqrt{4-y^2}} \int_{\frac{x^2+4y^2}{4}}^{\frac{48-x^2-4y^2}{8}} dz\, dxdy$$

$$= \int_0^2 \int_0^{2\sqrt{4-y^2}} z\Big]_{\frac{x^2+4y^2}{4}}^{\frac{48-x^2+4y^2}{8}} \mathrm{dx\, dy} = \int_0^2 \int_0^{2\sqrt{4-y^2}} \frac{3}{8}(16 - \mathrm{x}^2 - 4\mathrm{y}^2)\mathrm{dx\, dy}$$

Example 3 Use **triple integral** to find the volume in the first octant (الثمن) of the region that bounded by the surface $2z = 4 - x^2 - y^2$ and the plane xy.

Solution:

$$V = \iiint_D dV = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\frac{4-x^2-y^2}{2}} dz\, dydx$$

Example 4: Let D be the solid region that bounded above sphere (الكرة) $x^2 + y^2 + z^2 = 1$ and below $z^2 = x^2 + y^2$, find the value

$$V = \iiint_D zdV \ .$$

Solution:

$$V = \iiint_D z\, dV = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z\, dz\, dydx$$

Example 5: Use **triple integral** to find the volume the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.
Solution:

$$V = \iint_R \int_1^{5-x} dz\, dA = \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_1^{5-x} dz\, dxdy$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} z]_1^{5-x} \mathrm{dx\,dy} = \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (4-x)\mathrm{dx\,dy} = 4\int_0^3 \int_0^{\sqrt{9-y^2}} (4-x)\mathrm{dx\,dy} =$$

$$= 4\int_0^3 (4x - \frac{x^2}{2})\Big]_0^{\sqrt{9-y^2}} dy = 4\int_0^3 (4\sqrt{9-y^2} - \frac{9-y^2}{2})dy$$

Example 6: Use **triple integral** to find the volume of the solid enclosed between the xy- plane and $3z = 9 - x^2 - y^2$.

Solution:

$$\iiint_R \int_0^{\frac{9-x^2-y^2}{3}} dz\, dA = = \int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} z]_0^{\frac{9-x^2-y^2}{3}} \mathrm{dx\,dy} =$$

$$\frac{1}{3}\int_{-3}^{3} \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (9-x^2-y^2)\mathrm{dx\,dy} = \frac{4}{3}\int_0^3 \int_0^{\sqrt{9-y^2}} (9-x^2-y^2)\mathrm{dx\,dy} =$$

We can use polar coordinates

$$= \frac{4}{3}\int_0^{\frac{\pi}{2}} \int_0^3 (9-r^2)\mathrm{rdr\,d\theta} = -\frac{2}{3}\int_0^{\frac{\pi}{2}} (9-r^2)^2]_0^3 \mathrm{d\theta} = (27)\int_0^{\frac{\pi}{2}} \mathrm{d\theta} = (27)\ \theta]_0^{\frac{\pi}{2}}$$

$$= \frac{\mathbf{27}}{\mathbf{2}}\pi$$

Example 7: Use **triple integral** to find the volume bounded above surfaces

$z = 8 - x^2 - y^2$ and below $z = x^2 + 3y^2$.

Solution:

$8 - x^2 - y^2 = x^2 + 3y^2 \Rrightarrow 4y^2 = 8 - 2x^2 \Rrightarrow 2y^2 = 4 - x^2 \Rrightarrow y = \pm\sqrt{\frac{4-x^2}{2}}$

$$\int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz \text{ dy dx} = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} z]_{x^2+3y^2}^{8-x^2-y^2} \text{ dy dx}$$

$$= \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} 8 - x^2 - y^2 - (x^2 + 3y^2) \text{ dy dx} = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8 - 2x^2 + 4y^2) \text{dy dx}$$

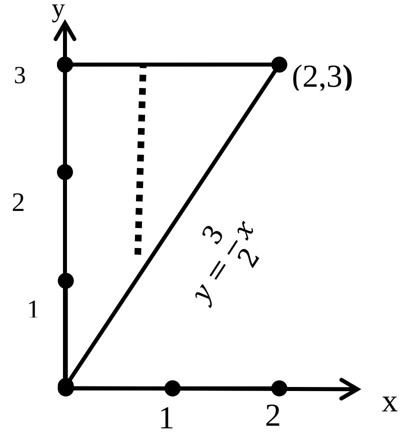
$$= \int_{-2}^{2} (8y - 2x^2 y + \frac{4y^3}{3})\Big]_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^{2} \left(8\sqrt{\frac{4-x^2}{2}} - 2x^2\sqrt{\frac{4-x^2}{2}} + \frac{4(\sqrt{\frac{4-x^2}{2}})^3}{3} \right) - \left(-8\sqrt{\frac{4-x^2}{2}} + 2x^2\sqrt{\frac{4-x^2}{2}} - \frac{4(\sqrt{\frac{4-x^2}{2}})^3}{3} \right) dx =$$

$$\int_{-2}^{2} \left(16\sqrt{\frac{4-x^2}{2}} - 4x^2\sqrt{\frac{4-x^2}{2}} + \frac{8\sqrt{\frac{4-x^2}{2}}^3}{3} \right) dx =$$

$$\int_{-2}^{2} \left(16\sqrt{\frac{4-x^2}{2}} - 4x^2\sqrt{\frac{4-x^2}{2}} + \frac{8\sqrt{\frac{4-x^2}{2}}^3}{3} \right) dx = \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4-x))^{\frac{3}{2}} dx = 8\sqrt{2}\pi$$

Example: Find the mass of triangular with vertices $(0,0),(0,3),(2,3)$, and the density at (x,y) is $\delta(x,y)=2x+y$



Solution:

$$\boldsymbol{M}=\iint_{\boldsymbol{R}}\delta(x,y)\boldsymbol{dA}$$

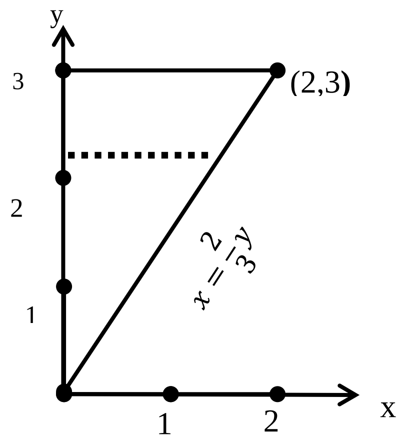
$$\boldsymbol{M}=\int_{\mathbf{0}}^{\mathbf{2}}\int_{\frac{3}{2}x}^{3}(2x+y)\boldsymbol{dydx}=\int_{\mathbf{0}}^{\mathbf{2}}\boldsymbol{2xy}+\frac{\boldsymbol{y}^{2}}{\mathbf{2}}\Big|_{\frac{3}{2}x}^{\mathbf{3}}\ \boldsymbol{dx}$$

$$=\int_{\mathbf{0}}^{\mathbf{2}}(\boldsymbol{6x}+\frac{\mathbf{9}}{\mathbf{2}}-\left[\boldsymbol{2x}\left(\frac{3}{2}x\right)+\frac{\frac{9}{4}x^{2}}{2}\right])\boldsymbol{dx}=\int_{\mathbf{0}}^{\mathbf{2}}(\boldsymbol{6x}+\frac{\mathbf{9}}{\mathbf{2}}-\left[\boldsymbol{3x^{2}}+\frac{9}{8}x^{2}\right])\boldsymbol{dx}$$

$$\int_{\mathbf{0}}^{\mathbf{2}}(\boldsymbol{6x}+\frac{\mathbf{9}}{\mathbf{2}}-\left[\frac{24}{8}\boldsymbol{x^{2}}+\frac{9}{8}x^{2}\right])\boldsymbol{dx}=\int_{\mathbf{0}}^{\mathbf{2}}(\boldsymbol{6x}+\frac{\mathbf{9}}{\mathbf{2}}-\frac{33}{8}\boldsymbol{x^{2}})\boldsymbol{dx}=\boldsymbol{3x^{2}}+\frac{\mathbf{9}}{\mathbf{2}}\boldsymbol{x}-\frac{33}{8(3)}\boldsymbol{x^{3}}\Big|_{\mathbf{0}}^{\mathbf{2}}$$

$$12+9-\frac{33}{8(3)}8=12+9-11=10$$

Reversing:



$$\boldsymbol{M}=\int_{\mathbf{0}}^{\mathbf{3}}\int_{0}^{\frac{2}{3}\mathrm{y}}(2x+y)\boldsymbol{dxdy}=\int_{\mathbf{0}}^{\mathbf{3}}\boldsymbol{x^{2}}+\boldsymbol{xy}\Big|_{0}^{\frac{2}{3}\mathrm{y}}\ \boldsymbol{dy}$$

$$=\int_{\mathbf{0}}^{\mathbf{3}}(\frac{4}{9}\ \mathrm{y}^{2}+\frac{2}{3}\ \mathrm{y}^{2})\boldsymbol{dy}=\int_{\mathbf{0}}^{\mathbf{3}}(\frac{4}{9}\ \mathrm{y}^{2}+\frac{6}{9}\ \mathrm{y}^{2})\boldsymbol{dy}=\int_{\mathbf{0}}^{\mathbf{3}}(\frac{10}{9}\ \mathrm{y}^{2})\boldsymbol{dy}$$

$$=\frac{10}{9(3)}\ \mathrm{y}^{3}\Big|_{\mathbf{0}}^{\mathbf{3}}=\frac{10}{9(3)}\ 27=10$$