# Laplace Transforms

1

## 1.1. INTRODUCTION

The Laplace transform* of a suitably defined function $f$ of a real variable $t$ is a related function F of a real variable $s$. The use of Laplace transforms provide a powerful method of solving differential and integral equations. The Laplace transform method also has the advantage that it solves initial value problems directly without first finding a general solution. The ready tables of Laplace transforms has reduced the problem of solving differential equations to merely algebraic manipulation.

## 1.2. LAPLACE TRANSFORM OF A FUNCTION

Let $f$ be a real valued function of the real variable $t$, defined for $t \geq 0$. Let $s$ be a real variable and consider the function F of $s$ defined by

$$\mathrm{F}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

for all values of $s$ for which this integral exists (that has some finite value). The function F($s$) defined by the integral $\int_0^\infty e^{-st} f(t)\, dt$ is called the **Laplace transform** of the function $f$ and is denoted by L($f$).

$$\therefore \qquad \mathbf{F(s) = L(f) = \int_0^\infty e^{-st}\, f(t)\, dt.}$$

**Remarks 1.** The original function $f$ depends on $t$ and its Laplace transform (F) depends on $s$.

**2**. Original functions are denoted by *lower case letters* and their Laplace transforms by the same letters in *capitals.*

**WORKING STEPS TO FIND LAPLACE TRANSFORMS OF f(t), t ≥ 0.**

**Step I.** Write $\mathrm{L}(f) = \int_0^\infty e^{-st} f(t)\, dt$.

**Step II.** Write this improper integral as $\lim\limits_{\mathrm{T} \to \infty} \int_0^{\mathrm{T}} e^{-st} f(t)\, dt$ and simplify $e^{-st} f(t)$.

**Step III.** Evaluate $\int_0^{\mathrm{T}} e^{-st} f(t)\, dt$.

**Step IV.** Find limit of $\int_0^{\mathrm{T}} e^{-st} f(t)\, dt$ as $\mathrm{T} \to \infty$. This gives the value of L($f$).

*__Pierre Simon Marquis De Laplace__ (1749—1827) French mathematician, made contribution to special functions, probability theory, potential theory and astronomy.

## 1.3. LAPLACE TRANSFORMS OF ELEMENTARY FUNCTIONS

In this section, we shall find the Laplace transforms of some elementary functions.

**Theorem. Prove that for t ≥ 0,**

**1.** $\mathbf{L(1) = \frac{1}{s}, s > 0}$

**2.** $\mathbf{L(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}, a > -1, s > 0}$

**3.** $\mathbf{L(t^n) = \frac{n!}{s^{n+1}}, n = 0, 1, 2, \ldots\ldots; s > 0}$

**4.** $\mathbf{L(e^{at}) = \frac{1}{s-a}, \ s > a}$

**5.** $\mathbf{L(\sinh at) = \frac{a}{s^2 - a^2}, \ s > |a|}$

**6.** $\mathbf{L(\cosh at) = \frac{s}{s^2 - a^2}, \ s > |a|}$

**7.** $\mathbf{L(\sin at) = \frac{a}{s^2 + a^2}, \ s > 0}$

**8.** $\mathbf{L(\cos at) = \frac{s}{s^2 + a^2}, \ s > 0}$.

**Proof.** By definition, the improper integeral $\int_k^\infty g(t)\, dt$ is equal to $\lim_{T \to \infty} \int_k^T g(t)\, dt$.

**1.** By definition, $L(1) = \int_0^\infty e^{-st} . 1\, dt$

$$\therefore \qquad L(1) = \lim_{T\to\infty} \int_0^T e^{-st}\, dt = \lim_{T\to\infty} \left[\frac{e^{-st}}{-s}\right]_0^T$$

$$= \frac{1}{s} \lim_{T\to\infty} \left[-\frac{1}{e^{st}}\right]_0^T = \frac{1}{s} \lim_{T\to\infty} \left[-\frac{1}{e^{sT}} + 1\right]$$

$$= \frac{1}{s}[0+1] = \frac{1}{s} \text{ for } s > 0.$$

$$\therefore \qquad \mathbf{L(1) = \frac{1}{s}, s > 0}.$$

**2.** By definition, $L(t^a) = \int_0^\infty e^{-st}\, t^a\, dt$.

Let $st = u$. $\therefore$ $s\, dt = du$.

$$\therefore \qquad L(t^a) = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^a \frac{du}{s} = \frac{1}{s^{a+1}} \int_0^\infty e^{-u}\, u^a\, du$$

$$= \frac{\Gamma(a+1)}{s^{a+1}}, \text{ provided } a > -1$$

$$\therefore \qquad \mathbf{L(t^a) = \frac{\Gamma(a+1)}{s^{a+1}}, a > -1, \quad s > 0}. \qquad (\because \quad t \ge 0, u \ge 0, u = st \Rightarrow s \ge 0)$$

**3.** By definition, $L(t^n) = \int_0^\infty e^{-st}\, t^n\, dt$.

Let $st = u$. $\therefore$ $s\, dt = du$

$\therefore \qquad \mathrm{L}(t^n) = \int_0^\infty e^{-u}\left(\frac{u}{s}\right)^n \frac{du}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-u}\, u^n\, du$

$= \frac{\Gamma(n+1)}{s^{n+1}}$, provided $n + 1 > 0$.

Let $\quad n = 0, 1, 2, \ldots\ldots \quad \therefore \quad \Gamma(n+1) = n\,!$

$\therefore \qquad \mathbf{L(t^n) = \frac{n\,!}{s^{n+1}}, n = 0, 1, 2, \ldots\ldots; s > 0.} \qquad (\because \quad t \geq 0, u \geq 0, u = st \Rightarrow s \geq 0)$

**4.** By definition, $\quad \mathrm{L}(e^{at}) = \int_0^\infty e^{-st}\, e^{at}\, dt$

$$= \lim_{\mathrm{T}\to\infty}\int_0^{\mathrm{T}} e^{(a-s)t}\, dt = \lim_{\mathrm{T}\to\infty}\left[\frac{e^{(a-s)t}}{a-s}\right]_0^{\mathrm{T}}$$

$$= \frac{1}{a-s}\lim_{\mathrm{T}\to\infty}\left[\frac{1}{e^{(s-a)\mathrm{T}}} - 1\right] = \frac{1}{a-s}[0-1] \text{ for } s > a \;= \frac{1}{s-a}$$

$\therefore \qquad \mathbf{L(e^{at}) = \frac{1}{s-a}, \; s > a.}$

**5.** $\qquad \mathrm{L}(\sinh at) = \mathrm{L}\left(\frac{e^{at} - e^{-at}}{2}\right) = \int_0^\infty e^{-st}\left(\frac{e^{at} - e^{-at}}{2}\right) dt$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s-a)t} - e^{-(s+a)t}\right) dt = \frac{1}{2}\lim_{\mathrm{T}\to\infty}\int_0^{\mathrm{T}} \left(e^{-(s-a)t} - e^{-(s+a)t}\right) dt$$

$$= \frac{1}{2}\lim_{\mathrm{T}\to\infty}\left[\frac{e^{-(s-a)t}}{a-s} + \frac{e^{-(s+a)t}}{s+a}\right]_0^{\mathrm{T}}$$

$$= \frac{1}{2}\lim_{\mathrm{T}\to\infty}\left[\frac{1}{a-s}\left(\frac{1}{e^{(s-a)\mathrm{T}}} - 1\right) + \frac{1}{s+a}\left(\frac{1}{e^{(s+a)\mathrm{T}}} - 1\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{a-s}(0-1) + \frac{1}{s+a}(0-1)\right] \text{ provided } s > a,\, s > -a$$

$$= \frac{1}{2}\left[\frac{2a}{s^2 - a^2}\right] = \frac{a}{s^2 - a^2}, \; s > |\, a\,|.$$

$\therefore \qquad \mathbf{L(\sinh at) = \frac{a}{s^2 - a^2}, \; s > |\, a\,|.}$

**7.** $\qquad \mathrm{L}(\sin at) = \int_0^\infty e^{-st} \sin at\, dt$

$\therefore \qquad \mathrm{L}(\sin at) = \lim_{\mathrm{T}\to\infty}\int_0^{\mathrm{T}} e^{-st} \sin at\, dt$

Let $\qquad \mathrm{I} = \int e^{-st} \sin at\, dt$

$\therefore \qquad \mathrm{I} = e^{-st}.\frac{-\cos at}{a} - \int -se^{-st}.\frac{-\cos at}{a}\, dt$

$$= -\frac{e^{-st}\cos at}{a} - \frac{s}{a}\int e^{-st}\cos at\,dt$$

$$= -\frac{e^{-st}\cos at}{a} - \frac{s}{a}\left[e^{-st}\frac{\sin at}{a} - \int - se^{-st}\,.\,\frac{\sin at}{a}\,dt\right]$$

$$\therefore \qquad \mathrm{I} = -\frac{e^{-st}}{a^2}[a\cos at + s\sin at] - \frac{s^2}{a^2}\mathrm{I}$$

$$\therefore \qquad \mathrm{I} = -\frac{e^{-st}}{s^2+a^2}[a\cos at + s\sin at]$$

$$\therefore \qquad \mathrm{L}(\sin at) = \lim_{\mathrm{T}\to\infty}\frac{-e^{-st}}{s^2+a^2}\Big[a\cos at + s\sin at\Big]_0^{\mathrm{T}}$$

$$= \frac{-1}{s^2+a^2}\lim_{\mathrm{T}\to\infty}\left[\frac{a\cos aT + s\sin aT}{e^{s\mathrm{T}}} - \frac{a+0}{1}\right]$$

$$= \frac{-1}{s^2+a^2}[0-a]\ , \text{ provided } s > 0$$

$$= \frac{a}{s^2+a^2}\ , \ s > 0.$$

$$\therefore \qquad \mathbf{L(\sin at)} = \frac{\mathbf{a}}{\mathbf{s^2+a^2}}\ , \ \mathbf{s > 0.}$$

**Remark.** The proofs of the parts 6 and 8 are left for readers as exercises.

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Find the Laplace transform of the following function :*

$$f(t) = \begin{cases} t+1, & 0 \le t \le 2 \\ 3, & t > 2. \end{cases}$$

**Sol.**

$$\mathrm{L}f(t) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st}(t+1)\,dt + \int_2^\infty e^{-st}\,.\,3\,dt$$

$$= \left[(t+1)\,.\,\frac{e^{-st}}{-s}\Bigg|_0^2 - \int_0^2 1\,.\,\frac{e^{-st}}{-s}\,dt\right] + 3\,.\,\frac{e^{-st}}{-s}\Bigg|_2^\infty$$

$$= -\frac{3}{s}e^{-2s} + \frac{1}{s} - \frac{e^{-st}}{s^2}\Bigg|_0^2 - \frac{3}{s}\left[\lim_{\mathrm{T}\to\infty} e^{-s\mathrm{T}} - e^{-2s}\right]$$

$$= -\frac{3e^{-2s}}{s} + \frac{1}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2} - \frac{3}{s}(0 - e^{-2s})$$

$$= -\frac{3e^{-2s}}{s} + \frac{1}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2} + \frac{3e^{-2s}}{s} = \frac{\mathbf{1}}{\mathbf{s^2}}[\mathbf{s} - \mathbf{e^{-2s}} + \mathbf{1}]\,.$$

**Example 2.** *Find the Laplace transform of the following function :*

$$f(t)=\begin{cases}0, & 0\le t<1\\ t, & 1<t<2\\ 1, & t>2.\end{cases}$$

**Sol.** $$\mathrm{L}(f(t))=\int_0^\infty e^{-st}\,f(t)\,dt$$

$$=\int_0^1 e^{-st}\,f(t)\,dt+\int_1^2 e^{-st}\,f(t)\,dt+\int_2^\infty e^{-st}\,f(t)\,dt$$

$$=\int_0^1 e^{-st}\,.\,0\,dt+\int_1^2 e^{-st}\,t\,dt+\int_2^\infty e^{-st}\,.\,1\,dt$$

$$=\int_0^1 0\,dt+\left[t\,.\,\frac{e^{-st}}{-s}\bigg|_1^2-\int_1^2 1\,.\,\frac{e^{-st}}{-s}\,dt\right]+\lim_{T\to\infty}\left[\frac{e^{-st}}{-s}\right]_2^T$$

$$=0-\frac{2}{se^{2s}}+\frac{1}{se^{s}}+\frac{e^{-st}}{-s^2}\bigg|_1^2-\frac{1}{s}\lim_{T\to\infty}\left[\frac{1}{e^{sT}}-\frac{1}{e^{2s}}\right]$$

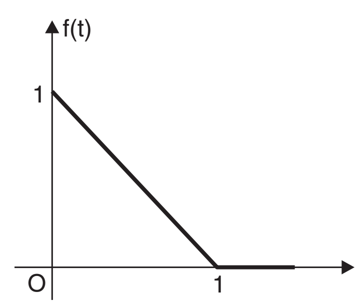
$$=-\frac{2}{se^{2s}}+\frac{1}{se^{s}}-\frac{1}{s^2}\left[\frac{1}{e^{2s}}-\frac{1}{e^{s}}\right]-\frac{1}{s}[0-1]\,,\quad s>0$$

$$=-\frac{2}{se^{2s}}+\frac{1}{se^{s}}-\frac{1}{s^2e^{2s}}+\frac{1}{s^2e^{s}}+\frac{1}{se^{2s}}\,,\quad s>0$$

$$=\frac{1}{s^2}[-se^{-2s}+se^{-s}-e^{-2s}+e^{-s}]\,,\quad s>0$$

$$=\mathbf{\frac{1}{s^2}(e^{-s}-e^{-2s})(s+1)\,,\quad s>0.}$$

**Example 3.** *Find the Laplace transform of the function shown in the graph.*




**Sol.** From the graph, we have

$$f(t)=\begin{cases}t\tan 135^\circ+1 & 0\le t\le 1\\ 0 & t\ge 1\end{cases}$$

or $$f(t)=\begin{cases}-t+1 & 0\le t\le 1\\ 0 & t\ge 1.\end{cases}$$

$$\therefore\ \mathrm{L}(f(t))=\int_0^\infty e^{-st}\,f(t)\,dt$$

$$=\int_0^1 e^{-st}\,f(t)\,dt+\int_1^\infty e^{-st}\,f(t)\,dt=\int_0^1 e^{-st}\,(-t+1)\,dt+\int_1^\infty e^{-st}\,.\,0\,dt$$

$$=\int_0^1 (1-t)\,e^{-st}\,dt+\int_0^\infty 0\,dt=(1-t)\,.\,\frac{e^{-st}}{-s}\bigg|_0^1-\int_0^1 -1\,.\,\frac{e^{-st}}{-s}\,dt+0$$

$$=0+\frac{e^0}{s}-\frac{e^{-st}}{-s^2}\bigg|_0^1=\frac{1}{s}+\frac{1}{s^2}\left[\frac{1}{e^s}-\frac{1}{e^0}\right]=\mathbf{\frac{1}{s}+\frac{1}{s^2}\left[\frac{1}{e^s}-1\right],\quad s\neq 0.}$$

**Example 4.** *Show that* $\int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

**Sol.** Let $\quad I = \int_0^\infty \cos x^2\, dx$.

$$t = x^2 \;\Rightarrow\; dt = 2x\,dx \;\Rightarrow\; dx = \frac{dt}{2\sqrt{t}}$$

$$x \to \infty \;\Rightarrow\; t \to \infty,\; x = 0 \;\Rightarrow\; t = (0)^2 = 0$$

$$\therefore \quad I = \int_0^\infty \frac{\cos t\, dt}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty \frac{e^{it} + e^{-it}}{2\sqrt{t}}\, dt$$

$$= \frac{1}{4}\left[\int_0^\infty e^{it}\,.\,t^{-1/2}\, dt + \int_0^\infty e^{-it}\,.\,t^{-1/2}\, dt\right]$$

$$= \frac{1}{4}\,[L(t^{-1/2})\,\big|_{s=-i} + L(t^{-1/2})\,\big|_{s=i}]$$

$$= \frac{1}{4}\left[\frac{\Gamma(1/2)}{s^{1/2}}\bigg|_{s=-i} + \frac{\Gamma(1/2)}{s^{1/2}}\bigg|_{s=i}\right]$$

$$= \frac{1}{4}\left[\frac{\sqrt{\pi}}{\sqrt{-i}} + \frac{\sqrt{\pi}}{\sqrt{i}}\right] = \frac{\sqrt{\pi}}{4}\left[\frac{1}{\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i} + \frac{1}{\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i}\right] \qquad \text{(Note this step)}$$

$$= \frac{\sqrt{2\pi}}{4}\left[\frac{1}{1-i} + \frac{1}{1+i}\right] = \frac{\sqrt{2\pi}}{4}\left[\frac{2}{2}\right] = \mathbf{\frac{1}{2}\sqrt{\frac{\pi}{2}}}.$$

## TEST YOUR KNOWLEDGE

**1.** By using standard formulae, write the Laplace transform of the following functions :

(*i*) $t^{3/2}$ (*ii*) $t^{5/3}$ (*iii*) $t^7$

(*iv*) $e^{3t}$ (*v*) $e^{-10t}$ (*vi*) $\sinh 5t$

(*vii*) $\cosh 7t$ (*viii*) $\sin 4t$ (*ix*) $\cos 6t$.

**2.** Find the Laplace transform of the function $f(t) = t,\ t \ge 0$.

**3.** Find the Laplace transform of the function $f(t) = 4t + 3,\ t \ge 0$.

**4.** Find the Laplace transform of the function $f(t) = \cosh at,\ t \ge 0$.

**5.** Find the Laplace transform of the function $f(t) = \cos at,\ t \ge 0$.

**6.** Find the Laplace transform of the following functions :

(*i*) $f(t) = \begin{cases} 4, & 0 \le t < 3 \\ 2, & t > 3 \end{cases}$ (*ii*) $f(t) = \begin{cases} t, & 0 \le t < 2 \\ 3, & t > 2 \end{cases}$.

**7.** Find the Laplace transform of the following function :

$$f(t) = \begin{cases} t & 0 \le t \le 1/2 \\ t - 1 & 1/2 < t \le 1 \\ 0 & t > 1 \end{cases}.$$