الاشتقاق الجزئي

Let $f(x,y)$ be defined in a region R of the xy-plan. If we think of y as fixed and x as variable, the derivative of $f(x,y)$ with respect to x is called the partial derivative with respect to x. This partial derivative is denoted by $\frac{\partial f}{\partial x}$ or f_x and defined by :

$$\frac{\partial f}{\partial x} = f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

If we write $z = f(x,y)$, the partial derivative is also denoted by $\frac{\partial z}{\partial x}$ or z_x .

Likewise, the partial derivative with respect to y, $\frac{\partial f}{\partial y}$ or $\frac{\partial z}{\partial y}$ or z_y , is the derivative of $f(x,y)$ with respect to y when x is regarded (يعتبر) as a constant and defined by :

$$z_y = \frac{\partial f}{\partial y} = f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

Example 1: If $z = x^2 y + e^{-xy^3}$

Solution: $\frac{\partial z}{\partial x} = 2xy - y^3 e^{-xy^3}$

$\frac{\partial z}{\partial y} = x^2 - 3xy^2 e^{-xy^3}$

Example 2: By using the definition of derivative, find f_x , f_y for the following:

1- $z = f(x,y) = x - y^2$ 2- $z = f(x,y) = \frac{x}{y}$

Solution: 1 the derivative with respect to x is المشتقة بالنسبة ل x هي

$$z_x = f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x + \Delta x - y^2 - (x - y^2)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{x + \Delta x - y^2 - x + y^2}{\Delta x} = \lim_{\Delta x \to 0} 1 = 1$$

$\therefore z_x = 1$ غاية الثابت يساوي الثابت نفسه

the derivative with respect to y is

$$z_y = f_y(x,y) = \lim_{\Delta y \to 0} \frac{x-(y+\Delta y)^2-(x-y^2)}{\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{x-y^2-2y\Delta y-(\Delta y)^2-x+y^2}{\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{\Delta y(-2y-\Delta y)}{\Delta y} = \lim_{\Delta y \to 0} -2y-\Delta y = -2y$$

$$\therefore z_y = -2y$$

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2 . $f(x,y) = \frac{x}{y}$

The derivative with respect to x is

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\frac{x+\Delta x}{y}-\frac{x}{y}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\frac{x+\Delta x-x}{y}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{y\Delta x} = \frac{1}{y}$$
$$\therefore f_x = \frac{1}{y}$$

Now, the derivative with respect to y is

$$z_y = f_y(x,y) = \lim_{\Delta y \to 0} \frac{\frac{x}{y+\Delta y}-\frac{x}{y}}{\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{\frac{xy-x(y+\Delta y)}{(y+\Delta y)y}}{\Delta y} = \lim_{\Delta y \to 0} \frac{xy-xy-x\Delta y}{y\Delta y(y+\Delta y)} = \lim_{\Delta y \to 0} \frac{-x}{y^2+y\Delta y}$$
$$= \frac{-x}{y^2+y(0)} = -\frac{x}{y^2}$$

$$\therefore z_y = -\frac{x}{y^2}$$

Example 3: Find z_x and z_y for the following

1) $z = x - y^2$, 2) $z = e^{xy}$, 3) $z = xe^{x^2y}$, 4) $z = x^2\sin(xy^2)$

Solution:1 the derivative as

1. $z_x = 1$, $z_y = -2y$

2. $z_x = ye^{xy}$, $z_y = xe^{xy}$,

3. $z_x = x(e^{x^2y}.2\mathrm{xy}) + e^{x^2y} = (2x^2y+1)e^{x^2y}$,
   $z_y = x^3e^{x^2y}$.

4. $z_x = \mathrm{x}^2(\cos(\mathrm{xy}^2).\mathrm{y}^2) + 2x\sin(\mathrm{xy}^2)$
   $\therefore z_x = \mathrm{x}^2\mathrm{y}^2\cos(\mathrm{xy}^2) + 2x\ \sin(\mathrm{xy}^2)$,

   $z_y = \mathrm{x}^2\cos(\mathrm{xy}^2).2xy$
   $\therefore z_y = 2x^3y\ cos(xy^2)$.

Example 4: If $\mathrm{z} = \frac{x}{y}$, prove that $(\mathrm{x}z_x + yz_y = 0)$

Solution: The derivative with respect to x and y are

$$\mathrm{z}_x = \frac{1}{y}, \quad \mathrm{z}_y = -\frac{x}{y^2}$$

$$\left[\mathrm{x}\left(\frac{1}{y}\right) + y\left(-\frac{x}{y^2}\right)\right] = \frac{x}{y} - \frac{x}{y} = 0$$

Example 5: If $\mathrm{w} = \mathrm{f(x,y,z)} = \mathrm{xy} + \mathrm{yz}^2 + xz$, find $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$

Solution: The derivative with respect to x , y and z are respectively as

$$\frac{\partial w}{\partial x} = w_x = y + z$$
$$\frac{\partial w}{\partial y} = w_y = x + \mathrm{z}^2$$
$$\frac{\partial w}{\partial z} = w_z = 2yz + \mathrm{x}$$

Example 6: If $\mathrm{w} = \mathrm{f(x,y,z)} = \mathrm{z}\sin(\mathrm{xy}^2 + 2z)$, find $\frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial w}{\partial z}$

Solution: The derivative with respect to x , y and z are respectively as

$$\frac{\partial w}{\partial x} = w_x = \mathrm{z}\cos(\mathrm{xy}^2 + 2z)\mathrm{y}^2 = \mathrm{y}^2\mathrm{z}\ \cos(\mathrm{xy}^2 + 2z),$$
$$\frac{\partial w}{\partial y} = w_y = \mathrm{z}\ \cos(\mathrm{xy}^2 + 2z)2\mathrm{xy} = 2\mathrm{xyz}\ \cos(\mathrm{xy}^2 + 2z)$$
$$\frac{\partial w}{\partial z} = w_z = z\ \cos(\ \mathrm{xy}^2 + 2z).2 + \sin(\mathrm{xy}^2 + 2z),$$

Higher order partial derivatives
المشتقات الجزئية ذوات الرتب العليا

If $z = f(x, y)$, then the four second partial derivatives are

$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial x^2} = f_{xx} = z_{xx}$	$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial y^2} = f_{yy} = z_{yy}$
$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y \partial x} = f_{xy} = z_{xy}$	$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial x \partial y} = f_{yx} = z_{yx}$

Example 1: If $f(x, y) = 3xy + y^2 + 2x$, find the four second partial derivatives i.e. $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}$.

Solution:

$$\frac{\partial f}{\partial x} = 3y + 2$$

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial x^2} = 0$$

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = \frac{\partial^2 f}{\partial y \partial x} = 3$$

$$\frac{\partial f}{\partial y} = 3x + 2y$$

$$\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] = \frac{\partial^2 f}{\partial x \partial y} = 3$$

Example 2: If $f(x, y) = x^4 + 2x^2y + 3xy^3$, find the four second partial derivatives i.e. $f_{xx}, f_{yy}, f_{xy}, f_{yx}$.

Solution:

$$f_x = 4x^3 + 4xy + 3y^3$$

$$f_{xx} = 12x^2 + 4y$$

$$f_{xy} = 4x + 9y^2$$

$$f_y = 2x^2 + 9xy^2$$

$$f_{yy} = 18xy$$

$$f_{yx} = 4x + 9y^2$$

Exercise 1 : Find the four second partial derivative i.e. $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ for the following

1- $\mathrm{f(x,y)} = x^2 \mathrm{y} e^{(x+y^2)}$, 2 - $\mathrm{f(x,y)} = 5x^2 + 18xy + x - 2y$,

Exercise 2 : If $\mathrm{w} = \mathrm{y}e^{x} + x\,lnz$, show that

1) $\mathrm{w}_{xz} = \mathrm{w}_{zx}$
2) $\mathrm{w}_{xzz} = \mathrm{w}_{zxz} = \mathrm{w}_{zzx}$

Exercise 3 : If $z = xye^{x+y}$, then find $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y \partial x}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^3 z}{\partial x \partial y \partial x}$.

Laplace Equation
(معادلة لابلاس)

Definition: A function $z = f(x, y)$ is called Harmonic function (دالة توافقية) iff satisfy the Laplace equation.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

If $w = f(x, y, z)$ be a function of three variables, then the Laplace equation has the form

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Example: Show that the following functions are Harmonic

1. $z = e^{-y} cosx$, 2. $z = e^{x} siny$

Solution:1. $z = e^{-y} cosx$

we must show this function is satisfy Laplace equation i.e.

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{\partial z}{\partial x} = -e^{-y} sinx \qquad \frac{\partial z}{\partial y} = -e^{-y} cosx$$

$$\frac{\partial^2 z}{\partial x^2} = -e^{-y} cos\,x \qquad \frac{\partial^2 z}{\partial y^2} = e^{-y} cosx$$

$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = -e^{-y} cos\,x + e^{-y} cosx = 0$$

Its satisfied Laplace equation.

2. $z = e^x siny$

$$\frac{\partial z}{\partial x} = e^x siny$$
$$\frac{\partial^2 z}{\partial x^2} = e^x siny$$

$$\frac{\partial z}{\partial y} = e^x cosy$$
$$\frac{\partial^2 z}{\partial y^2} = -e^x siny$$

$$\therefore \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = e^x siny + -e^x siny = 0$$

Its satisfied Laplace equation.

Example: if $w = e^{(3x+4y)}\ \sin(5z)$, Prove that it satisfied Laplace equation

Solution: we must show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$$

$$\frac{\partial w}{\partial x} = 3e^{(3x+4y)}\ \sin(5z)$$
$$\boxed{\frac{\partial^2 w}{\partial x^2} = 9e^{(3x+4y)}\ \sin(5z)}$$

$$\frac{\partial w}{\partial z} = 5\ e^{(3x+4y)}\ \cos(5z)$$
$$\boxed{\frac{\partial^2 w}{\partial z^2} = -25\ \ e^{(3x+4y)}\ \sin(5z)}$$

$$\frac{\partial w}{\partial y} = 4\ \ e^{(3x+4y)}\ \sin(5z)$$
$$\boxed{\frac{\partial^2 w}{\partial y^2} = 16\ \ e^{(3x+4y)}\ \sin(5z)}$$

$$\therefore 9\ e^{(3x+4y)}\ \sin(5z) + 16\ e^{(3x+4y)}\ \sin(5z) - 25\ e^{(3x+4y)}\ \sin(5z) = 0$$

Its satisfied Laplace equation.

Exercise : Show that the following functions are satisfied Laplace equation

1. $f(x,y) = \ln(4x^2 + 4y^2)$,
2. $f(x,y) = e^{-y}\ sinx$
3. $f(x,y) = x^2 - y^2$
4. $f(x,y) = x^2 + y^2 - 2z^2$

Exercise : If $f(x,y) = \ln(x-y)$,

1.find the four second partial derivatives

2. prove that $f_{xy} = f_{yx}$

3. Is the above function satisfied Laplace ?