$= \delta x_i$

$\leq \|P\| < \delta$

It follows, from (1), that

$|f(x) - f(y)| < \frac{\varepsilon}{2(b-a)}, \qquad \forall\, x, y \in [x_{i-1}, x_i]$

$\Rightarrow \sup_{x\in[x_{i-1}, x_i]} f(x) - \inf_{x\in[x_{i-1}, x_i]} f(x) \leq \frac{\varepsilon}{2(b-a)}$

$\Rightarrow M_i - m_i \leq \frac{\varepsilon}{2(b-a)}$

Then

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} M_i\delta x_i - \sum_{i=1}^{n} m_i\delta x_i$$

$$= \sum_{i=1}^{n} (M_i - m_i)\delta x_i$$

$$\leq \frac{\varepsilon}{2(b-a)} \sum_{i=1}^{n} \delta x_i$$

$$= \frac{\varepsilon}{2(b-a)} \times (b-a) = \frac{1}{2}\varepsilon < \varepsilon$$

By Theorem (2.3), f is Riemann integrable on $[a, b]$.

Theorem (2.5): Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, then f Riemann integrable.

Theorem (2.6): Let f be a <u>bounded</u> function on $[a, b]$.If f is <u>monotonic</u> on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Proof:

Assume that

f is increasing on $[a, b]$;   (If f is decreasing, the proof is similar).

If f is constant on $[a, b]$ $\Rightarrow$ f is Riemann integrable on $[a, b]$.

So, we assume that f is non-constant on $[a, b]$;

$\Rightarrow$ $f(a) \neq f(b)$.

For any given $\varepsilon > 0$, let P be any partition of $[a, b]$ with mesh $\|P\|$ such that $\|P\| < \frac{\varepsilon}{f(b)-f(a)}$. Then,

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{n} M_i \delta x_i - \sum_{i=1}^{n} m_i \delta x_i \\
&= \sum_{i=1}^{n} (M_i - m_i) \delta x_i \\
&= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \delta x_i \quad \text{(since } f \text{ is increasing on } [a, b]\text{)} \\
&\leq ||P|| \times \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\
&= \|P\| \times \big(f(b) - f(a)\big) < \varepsilon
\end{aligned}$$

By Theorem (2.3), f is Riemann integrable on $[a, b]$.

Proposition (2.1): Any <u>nondecreasing</u> function f on $[a, b]$ is Riemann integrable.

Proof:

Since $f(a) \leq f(x) \leq f(b)$, $\forall\, x \in [a, b]$

$\Rightarrow f$ is bounded on $[a, b]$.

For any given $\varepsilon > 0$, let P be any partition of $[a, b]$ with mesh $\|P\|$ such that $\|P\| < \frac{\varepsilon}{f(b)-f(a)}$. Then,

$$
\begin{aligned}
U(f,P) - L(f,P) &= \sum_{i=1}^{n} M_i \delta x_i - \sum_{i=1}^{n} m_i \delta x_i \\
&= \sum_{i=1}^{n} (M_i - m_i) \delta x_i \\
&= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \delta x_i \\
&\leq ||P|| \times \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \\
&= \|P\| \times \big(f(b) - f(a)\big) < \varepsilon \\
&\leq \tfrac{\varepsilon}{f(b)-f(a)} \times \big(f(b) - f(a)\big) < \varepsilon
\end{aligned}
$$

By Theorem (2.3), f is Riemann integrable on $[a, b]$.

Theorem (2.7): If $f \in \mathcal{R}[a,b]$ and $g \in \mathcal{R}[a,b]$, then $f + g \in \mathcal{R}[a,b]$, $fg \in \mathcal{R}[a,b]$ and $cf \in \mathcal{R}[a,b]$ for every constant c.

Corollary (2.1): Suppose that f is Riemann integrable on $[a, b]$. Then $|f|$ is also Riemann integrable.

Corollary (2.2): If $f: [a, b] \to R$ is a Riemann integrable, then

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

Proof: We know that

$f(x) \leq |f(x)|$ and $-f(x) \leq |f(x)|$, $\forall \ x \in [a, b]$

Hence

$$\int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \quad \text{and} \quad -\int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$

$$\Rightarrow \quad \left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

Corollary (2.3): Suppose that $f:[a,b] \to R$ is a Riemann integrable. We set

$$m = \inf_{x\in[a,b]} f(x), \qquad M = \sup_{x\in[a,b]} f(x)$$

Then $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

Proof:

We have $m \le f(x) \le M, \ \forall \ x \in [a,b]$

So that

$$m(b-a) = \int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx \le M(b-a)$$

$\Rightarrow \quad m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

Proposition (2.2): Suppose that $f:[a,b] \to R$ is Riemann integrable and $f(x) \ge 0$ for any $x \in [a,b]$. Then

$$\int_a^b f(x)\,dx \ge 0\,.$$

Exercises (2.2): (Homework)

(1) Give an example of a function f on $[0,1]$ that is not Riemann integrable for which $|f|$ is Riemann integrable.

(2) Show $|\int_{-2\pi}^{2\pi} x^2 \, sin^8(e^x)dx| \le \frac{16\pi^3}{3}$.