TEST YOUR KNOWLEDGE

Find the inverse Laplace transform of the following functions by the method of partial fractions (Q No 1–12) :

1. (*i*) $\dfrac{1}{(s-1)(s-2)}$ (*ii*) $\dfrac{1}{(s+a)(s+b)}$

2. (*i*) $\dfrac{3s+7}{s^2-2s-3}$ (*ii*) $\dfrac{3s-11}{s^2-7s+12}$

3. (*i*) $\dfrac{2s^2+5s-4}{s^3+s^2-2s}$ (*ii*) $\dfrac{s^2-6}{s^3+4s^2+3s}$

4. (*i*) $\dfrac{1}{s^2(s^2+1)}$ (*ii*) $\dfrac{1}{s^2(s^2+1)(s^2+9)}$

5. (*i*) $\dfrac{s^2-2s+3}{(s-1)^2(s+1)}$ (*ii*) $\dfrac{1}{(s+2)^2(s-2)}$

6. (*i*) $\dfrac{5s-2}{s^2(s+2)(s-1)}$ (*ii*) $\dfrac{1}{s^2(s-a)^2}$

7. (*i*) $\dfrac{5s^2-15s-11}{(s+1)(s-2)^3}$ (*ii*) $\dfrac{3s^3-3s^2-40s+36}{(s^2-4)^2}$

8. (*i*) $\dfrac{1}{(s+1)(s^2+1)}$ (*ii*) $\dfrac{5s+3}{(s-1)(s^2+2s+5)}$

9. (*i*) $\dfrac{3s+1}{(s-1)(s^2+1)}$ (*ii*) $\dfrac{1}{s^3(s^2+1)}$

10. (*i*) $\dfrac{s}{s^4+s^2+1}$ (*ii*) $\dfrac{1}{s^3-a^3}$

11. (*i*) $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$ (*ii*) $\dfrac{s^3}{s^4-a^4}$

12. (*i*) $\dfrac{s}{(s+1)^2(s^2+1)}$ (*ii*) $\dfrac{a(s^2-2a^2)}{s^4+4a^4}$.

Answers

1. (*i*) $e^{2t}-e^{t}$ (*ii*) $\dfrac{e^{-at}-e^{-bt}}{b-a}$

2. (*i*) $4e^{3t}-e^{-t}$ (*ii*) $e^{4t}+2e^{3t}$

3. (*i*) $2+e^{t}-e^{-2t}$ (*ii*) $-2+\dfrac{5}{2}e^{-t}+\dfrac{1}{2}e^{-3t}$

4. (*i*) $t-\sin t$ (*ii*) $\dfrac{1}{9}t-\dfrac{1}{8}\sin t+\dfrac{1}{216}\sin 3t$

5. (*i*) $\left(t-\dfrac{1}{2}\right)e^{t}+\dfrac{3}{2}e^{-t}$ (*ii*) $\dfrac{1}{16}(e^{2t}-(4t+1)e^{-2t})$

6. (*i*) $t + e^t + e^{-2t} - 2$ (*ii*) $\frac{1}{a^3}[(2 + at) - (2 - at)e^{at}]$

7. (*i*) $e^{2t}\left[\frac{1}{3} + 4t - \frac{7}{2}t^2\right] - \frac{1}{3}e^{-t}$ (*ii*) $-10t \sinh 2t + 3e^{-2t} + 3te^{2t}$

8. (*i*) $\frac{1}{2}[e^{-t} - \cos t + \sin t]$ (*ii*) $e^t - e^{-t}\left(\cos 2t - \frac{3}{2}\sin 2t\right)$

9. (*i*) $2e^t - 2\cos t + \sin t$ (*ii*) $\frac{1}{2}t^2 + \cos t - 1$

10. (*i*) $\frac{2}{\sqrt{3}} \sin \frac{t\sqrt{3}}{2} \sinh \frac{t}{2}$

(*ii*) $\frac{1}{3a^2}\left[e^{at} - e^{-at/2}\left(\cos\left(\frac{\sqrt{3}}{2}at\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{2}at\right)\right)\right]$

11. (*i*) $\frac{1}{5}(1 + e^{-t})\sin t + \frac{3}{5}(1 - e^{-t})\cos t$ (*ii*) $\frac{1}{2}[\cos at + \cosh at]$

12. (*i*) $\frac{1}{2}[\sin t - te^{-t}]$ (*ii*) $\cos at \sinh at$.

Hints

4. (*i*) $\frac{1}{s^2(s^2+1)} = \frac{1}{z(z+1)}$, where $z = s^2$

$$= \frac{1}{z(1)} + \frac{1}{(-1)(z+1)} = \frac{1}{z} - \frac{1}{z+1} = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

10. (*i*) $s^4 + s^2 + 1 = (s^4 + 2s^2 + 1) - s^2$.

2.14. SOLUTION OF DIFFERENTIAL EQUATIONS BY USING LAPLACE TRANSFORMATION

We know that a differential equation with initial conditions is solved by first finding the general solution of the given equation and then finding the values of the arbitrary constants by using the given initial conditions. By using Laplace transformations, we find the required solution without first finding the general solution of the given equation.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following equations :*

(*i*) $\frac{d^2y}{dt^2} + y = 6\cos 2t$, *where* $y'(0) = 1$, $y(0) = 3$.

(*ii*) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$, *where* $y = \frac{dy}{dt} = 0$, *when* $t = 0$.

(*iii*) $(D^2 + m^2)\, y = a \cos nt$, *where* $y = 0 = \frac{dy}{dt}$ *when* $t = 0$.

(*iv*) $\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$, *where* $y = 1$, $\frac{dy}{dt} = 2$, $\frac{d^2y}{dt^2} = 2$ at $t = 0$.

Sol. (*i*) We have

$$\frac{d^2y}{dt^2} + y = 6 \cos 2t.$$

$$\Rightarrow \qquad y'' + y = 6 \cos 2t$$

Taking Laplace Transform, we have

$$L(y'') + L(y) = 6L(\cos 2t)$$

$$\Rightarrow \qquad [s^2L(y) - s \,.\, y(0) - y'(0)] + L(y) = 6 \frac{s}{s^2 + (2)^2}$$

$$\Rightarrow \qquad s^2 \bar{y} - s \,.\, 3 - 1 + \bar{y} = \frac{6s}{s^2 + 4} \qquad \text{(Putting } \bar{y} = L(y))$$

$$\Rightarrow \qquad (s^2 + 1)\, \bar{y} = \frac{6s}{s^2 + 4} + 3s + 1$$

$$\Rightarrow \qquad \bar{y} = \frac{6s}{(s^2 + 1)(s^2 + 4)} + \frac{3s}{s^2 + 1} + \frac{1}{s^2 + 1}$$

Taking inverse Laplace transform, we have

$$y = 6L^{-1}\left(\frac{s}{(s^2 + 1)(s^2 + 4)}\right) + 3L^{-1}\left(\frac{s}{s^2 + 1}\right) + L^{-1}\left(\frac{1}{s^2 + 1}\right) \qquad ...(1)$$

$$\frac{s}{(s^2 + 1)(s^2 + 4)} = \frac{s}{s^2 + 1} \cdot \frac{1}{s^2 + 4} = F(s)\, G(s),$$

where $\qquad F(s) = \frac{s}{s^2 + 1} \quad$ and $\quad G(s) = \frac{1}{s^2 + 4}.$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{s}{s^2 + 1}\right) = \cos t = f(t), \text{ say}$$

and $\qquad L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s^2 + 4}\right) = \frac{1}{2} \sin 2t = g(t), \text{ say}$

By **convolution theorem,**

$$L^{-1}(F(s)\, G(s)) = f * g = \int_0^t f(T)\, g(t - T)\, dT$$

$$= \int_0^t (\cos T)\left(\frac{1}{2} \sin 2(t - T)\right) dT = \frac{1}{4} \int_0^t 2 \sin\,(2t - 2T) \cos T\, dT$$

$$= \frac{1}{4} \int_0^t (\sin\,(2t - T) + \sin\,(2t - 3T))\, dT$$

$$= \frac{1}{4}\left[-\frac{\cos\,(2t - T)}{-1} - \frac{\cos\,(2t - 3T)}{-3}\right]_0^t$$

$$= \frac{1}{4}\left[\cos t - \cos 2t + \frac{\cos t}{3} - \frac{\cos 2t}{3}\right]$$

$$= \frac{1}{4}\left[\frac{4}{3} \cos t - \frac{4}{3} \cos 2t\right] = \frac{1}{3}\ [\cos t - \cos 2t]$$

Also, $\quad L^{-1}\left(\dfrac{s}{s^2+1}\right) = \cos t \quad$ and $\quad L^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t$

$\therefore \quad (1) \Rightarrow \qquad y = 6 \cdot \dfrac{1}{3}(\cos t - \cos 2t) + 3\cos t + \sin t$

or $\qquad y = \mathbf{5 \cos t + \sin t - 2 \cos 2t.}$

(*ii*) We have

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t.$$

$$\Rightarrow \qquad y'' + 2y' - 3y = \sin t$$

Taking Laplace transform, we have

$$L(y'') + 2L(y') - 3L(y) = L(\sin t)$$

$$\Rightarrow \qquad [s^2L(y) - s \cdot y(0) - y'(0)] + 2[sL(y) - y(0)] - 3L(y) = \frac{1}{s^2+1}$$

$$\Rightarrow \qquad s^2\bar{y} - s.0 - 0 + 2[s\bar{y} - 0] - 3\bar{y} = \frac{1}{s^2+1}$$

$$\Rightarrow \qquad (s^2 + 2s - 3)\,\bar{y} = \frac{1}{s^2+1}$$

$$\Rightarrow \qquad \bar{y} = \frac{1}{(s^2+1)(s^2+2s-3)} = \frac{1}{(s^2+1)(s+3)(s-1)}$$

Taking inverse Laplace transform, we have

$$y = L^{-1}\left(\frac{1}{(s^2+1)(s+3)(s-1)}\right)$$

Let $\quad \dfrac{1}{(s^2+1)(s+3)(s-1)} = \dfrac{As+B}{s^2+1} + \dfrac{C}{s+3} + \dfrac{D}{s-1}$

Multiplying by $(s^2 + 1)(s + 3)(s - 1)$, we get

$$1 = (As + B)(s + 3)(s - 1) + C(s^2 + 1)(s - 1) + D(s^2 + 1)(s + 3) \qquad ...(1)$$

$s = -3 \Rightarrow \quad 1 = 0 + c(10)(-4) + 0 \qquad \Rightarrow \quad C = -\dfrac{1}{40}$

$s = 1 \quad \Rightarrow \quad 1 = 0 + 0 + D(2)(4) \qquad \Rightarrow \quad D = \dfrac{1}{8}$

Comparing the coefficients of s^3 and constant terms in (1), we get

$$0 = A + C + D \qquad ...(2)$$

$$1 = -3B - C + 3D \qquad ...(3)$$

$(2) \quad \Rightarrow \qquad A = -C - D = \dfrac{1}{40} - \dfrac{1}{8} = -\dfrac{1}{10}$

$(3) \quad \Rightarrow \qquad 3B = -1 - C + 3D = -1 + \dfrac{1}{40} + \dfrac{3}{8} = -\dfrac{3}{5} \quad \therefore \quad B = -\dfrac{1}{5}$

$$\frac{1}{(s^2+1)(s+3)(s-1)} = \frac{-\frac{1}{10}s - \frac{1}{5}}{s^2+1} + \frac{-\frac{1}{40}}{s+3} + \frac{\frac{1}{8}}{s-1}$$

$$= -\frac{1}{10}\cdot\frac{s}{s^2+1} - \frac{1}{5}\cdot\frac{1}{s^2+1} - \frac{1}{40}\cdot\frac{1}{s+3} + \frac{1}{8}\cdot\frac{1}{s-1}$$

$$\therefore \qquad y = -\frac{1}{10}L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{5}L^{-1}\left(\frac{1}{s^2+1}\right) - \frac{1}{40}L^{-1}\left(\frac{1}{s+3}\right) + \frac{1}{8}L^{-1}\left(\frac{1}{s-1}\right)$$

$$= \mathbf{-\frac{1}{10}\cos t - \frac{1}{5}\sin t - \frac{1}{40}e^{-3t} + \frac{1}{8}e^{t}.}$$

(*iii*) We have $(D^2 + m^2)\, y = a \cos nt$.

$$\Rightarrow \qquad y'' + m^2 y = a \cos nt$$

Taking Laplace transform, we have

$$L(y'') + m^2 L(y) = a\, L(\cos nt)$$

$$\Rightarrow \qquad [s^2 L(y) - s\,.\,y(0) - y'(0)] + m^2 L(y) = a\,.\,\frac{s}{s^2+n^2}$$

$$\Rightarrow \qquad s^2\bar{y} - s.0 - 0 + m^2\bar{y} = \frac{as}{s^2+n^2}$$

$$\Rightarrow \qquad (s^2+m^2)\,\bar{y} = \frac{as}{s^2+n^2}$$

$$\Rightarrow \qquad \bar{y} = \frac{as}{(s^2+m^2)(s^2+n^2)}$$

$$\therefore \qquad y = aL^{-1}\left(\frac{s}{(s^2+m^2)(s^2+n^2)}\right) \qquad \text{...(1)}$$

$$\frac{s}{(s^2+m^2)(s^2+n^2)} = \frac{s}{s^2+m^2}\cdot\frac{1}{s^2+n^2} = F(s)\, G(s), \quad \text{where } F(s) = \frac{s}{s^2+m^2}$$

and $$G(s) = \frac{1}{s^2+n^2}.$$

$$L^{-1}(F(s)) = L^{-1}\left(\frac{s}{s^2+m^2}\right) = \cos mt = f(t), \text{ say}$$

and $$L^{-1}(G(s)) = L^{-1}\left(\frac{1}{s^2+n^2}\right) = \frac{1}{n}\sin nt = g(t), \text{ say}$$

By **convolution theorem,**

$$L^{-1}(F(s)\, G(s)) = f * g = \int_0^t f(T)\, g(t-T)\, dT$$

$$= \int_0^t \cos mT\,.\,\frac{1}{n}\sin n(t-T)\, dT$$

$$= \frac{1}{2n}\int_0^t 2\sin(nt - nT)\cos mT\, dT$$

$$= \frac{1}{2n}\int_0^t [\sin(nt + (m-n)T) + \sin(nt - (m+n)T)]\, dT$$

$$= \frac{1}{2n}\left[-\frac{\cos(nt+(m-n)T)}{m-n} - \frac{\cos(nt-(m+n)T)}{-(m+n)}\right]_0^t$$

$$= \frac{1}{2n}\left[-\frac{\cos mt}{m-n} + \frac{\cos nt}{m-n} + \frac{\cos mt}{m+n} - \frac{\cos nt}{m+n}\right]$$

$$= \frac{1}{2n}\left[\left(\frac{1}{m+n} - \frac{1}{m-n}\right)\cos mt + \left(\frac{1}{m-n} - \frac{1}{m+n}\right)\cos nt\right]$$

$$= \frac{1}{2n}\left[-\frac{2n\cos mt}{m^2-n^2} + \frac{2n\cos nt}{m^2-n^2}\right] = \frac{1}{m^2-n^2}(\cos nt - \cos mt)$$

$$\therefore \quad (1) \quad \Rightarrow \qquad y = \frac{\mathbf{a}}{\mathbf{m^2 - n^2}}\ \mathbf{(\cos nt - \cos mt).}$$

(*iv*) We have

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0.$$

$$\Rightarrow \qquad y''' + 2y'' - y' - 2y = 0$$

Taking Laplace transform, we have

$$L(y''') + 2L(y'') - L(y') - 2L(y) = 0$$

$$\Rightarrow \qquad [s^3L(y) - s^2 . y(0) - s . y'(0) - y''(0)] + 2[s^2L(y) - s . y(0) - y'(0)] - [sL(y) - y(0)] - 2L(y) = 0$$

$$\Rightarrow \qquad (s^3\bar{y} - s^2.1 - s.2 - 2) + 2(s^2\bar{y} - s.1 - 2) - (s\bar{y} - 1) - 2\bar{y} = 0$$

$$\Rightarrow \qquad s^3\bar{y} - s^2 - 2s - 2 + 2s^2\bar{y} - 2s - 4 - s\bar{y} + 1 - 2\bar{y} = 0$$

$$\Rightarrow \qquad (s^3 + 2s^2 - s - 2)\ \bar{y} = s^2 + 4s + 5$$

$$\bar{y} = \frac{s^2+4s+5}{s^3+2s^2-s-2} = \frac{s^2+4s+5}{s^2(s+2)-(s+2)}$$

$$\therefore \qquad \bar{y} = \frac{s^2+4s+5}{(s-1)(s+1)(s+2)}$$

$$= \frac{1+4+5}{(s-1)(2)(3)} + \frac{1-4+5}{(-2)(s+1)(1)} + \frac{4-8+5}{(-3)(-1)(s+2)}$$

$$= \frac{5}{3}\cdot\frac{1}{s-1} - \frac{1}{s+1} + \frac{1}{4}\cdot\frac{1}{s+2}$$

$$\therefore \qquad y = \frac{5}{3}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{3}L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \mathbf{\frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}.}$$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\dfrac{d^2y}{dt^2} + y = t$, where $y(0) = 1$, $y'(0) = -2$.

2. $\dfrac{d^2y}{dt^2} + 4\dfrac{dy}{dt} + 3y = e^{-t}$, where $y(0) = y'(0) = 1$.

3. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} = 2$, where $x(0) = 3$, $x'(0) = 1$.

4. $y'' - 3y' + 2y = 4t + e^{3t}$, where $y(0) = 1$, $y'(0) = -1$.

5. $\dfrac{d^3y}{dt^3} - 3\dfrac{d^2y}{dt^2} + 3\dfrac{dy}{dt} - y = t^2e^t$, where $y(0) = 1$, $y'(0) = 0$, $y''(0) = -2$.

6. $(D^4 + 2D^2 + 1)\, y = 0$, where $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$, $y'''(0) = -3$.

7. $\dfrac{d^2y}{dt^2} + \dfrac{dy}{dt} = t^2 + 2t$, where $y(0) = 4$, $y'(0) = -2$.

Answers

1. $y = t - 3\sin t + \cos t$

2. $y = -\dfrac{3}{4}e^{-3t} + \dfrac{7}{4}e^{-t} + \dfrac{1}{2}te^{-t}$

3. $x = 2t + e^{-t} + 2$

4. $y = 2t + 3 + \dfrac{1}{2}e^{3t} - \dfrac{1}{2}e^{t} - 2e^{2t}$

5. $y = e^t\left[\dfrac{t^5}{60} - \dfrac{t^2}{2} - t + 1\right]$

6. $y = t(\sin t + \cos t)$

7. $y = \dfrac{1}{3}t^3 + 2e^{-t} + 2$.