$$\|T(f)-T(f_0)\| \leq \|f-f_0\| \max_{a\leq t\leq b} |f(t)+f_0(t)|$$

$$< \delta \max_{a\leq t\leq b} \{|f(t)-f_0(t)+f_0(t)+f_0(t)|\}$$

$$\leq \delta \max_{a\leq t\leq b} \{|f(t)-f_0(t)| + |2f_0(t)|\}$$

$$= \delta(\max_{a\leq t\leq b} |f(t)-f_0(t)| + 2\max_{a\leq t\leq b} |f_0(t)|)$$

$$< \delta(\delta + 2M) \quad \text{from } ①$$

$$\leq \delta(1+2M)$$

$$\leq \frac{\epsilon}{1+2M}(1+2M) \longrightarrow \text{from } ②$$

$$= \epsilon$$

$\therefore$ T is continuous at f_0

Since f_0 was arbitrary it follow that

T is continuous on $C[a,b]$

Example Any Norm $\|\cdot\|: X \longrightarrow \mathbb{R}$ on vector space X over K is continuous on X

Solution Let $x_0 \in X$ is fixed with

$\|x-x_0\| < \delta$ for all $x \in X$ take $\delta = \epsilon$

$\because \big|\|x\| - \|x_0\|\big| < \|x-x_0\| < \delta = \epsilon$

$\therefore \big|\|x\| - \|x_0\|\big| < \epsilon \quad \forall x \in X$

$\therefore \|\cdot\|$ is continuous at x_0, since x_0 was arbitrary

so $\|\cdot\|$ is continuous at X.

31 1 **Lemma** 2 The addition operator

$+ : X \times X \longrightarrow X$ when $(X, \|\cdot\|)$ is normed

when $(X \times X, \|\cdot\|_2)$ also normed

$\|(x,y)\|_2 = \|x\| + \|y\|$

proof: Let $z_0 = (x_0, y_0) \in X \times X$ be fixed

such for $z = (x,y)$ and any $\delta > 0$ $\|z - z_0\|_2 < \delta$

$$\begin{aligned} \|z - z_0\|_2 &= \|(x,y) - (x_0, y_0)\| \\ &= \|(x - x_0, y - y_0)\| \\ &= \|x - x_0\| + \|y - y_0\| < \delta \end{aligned}$$

we take $\delta = \epsilon$

$\cancel{+(x,y)}$

$$\begin{aligned} \|+(z) - +(z_0)\| &= \|+(x,y) - +(x_0,y_0)\| \\ &= \|x + y - (x_0 + y_0)\| \\ &= \|x - x_0 + y - y_0\| \\ &\leq \|x - x_0\| + \|y - y_0\| \\ &< \delta = \epsilon \end{aligned}$$

$\therefore$ $+$ is continuous at $z_0 = (x_0, y_0)$

since z_0 is an arbiturary on $X \times X$

So $+$ is continuous at $X \times X$.

Lemma

The scalar multiplication operator

$\cdot : K \times X \longrightarrow X$

where $(X, \|\cdot\|)$ is normed space

Let $(\alpha_n)_{n=1}^{\infty} \in K$ such that $\alpha_n \longrightarrow \alpha$

$\forall \epsilon > 0$, $\exists\, k_1 \in N$ such that

$$|\alpha_n - \alpha| < \frac{\epsilon}{2\|x_0\|}$$

where $x_0 \in X$

Since every convergent sequence is bounded $\exists M > 0$ such that

$$|\alpha_n| < M \qquad \forall n \in N$$

Let $(x_n)_{n=1}^{\infty}$ sequence in X such that

$$x_n \longrightarrow x_0$$

$\forall \epsilon > 0$, $\exists\, k_2 \in N$ such that

$$\|x_n - x_0\| < \frac{\epsilon}{2M} \qquad \forall n \in N$$

$$\|\cdot(\alpha_n, x_n) - \cdot(\alpha, x_0)\| = \|\alpha_n x_n - \alpha x_0\|$$

$$= \|\alpha_n x_n - \alpha_n x_0 + \alpha_n x_0 - \alpha x_0\|$$

$$= \|\alpha_n(x_n - x_0) + x_0(\alpha_n - \alpha)\|$$

$$\leq \|\alpha_n(x_n - x_0)\| + \|x_0(\alpha_n - \alpha)\|$$

(21) . (3)

$$= \|\alpha_n (x_n - x_0)\| + \|x_0(\alpha_n - \alpha_0)\|$$

$$= |\alpha_n| \, \|x_n - x_0\| + |\alpha_n - \alpha_0| \, \|x_0\|$$

$$\leq M \, \|x_n - x_0\| + |\alpha_n - \alpha_0| \, \|x_0\|$$

$$< \cancel{M} \frac{\epsilon}{2\cancel{M}} + \frac{\epsilon}{2\cancel{\|x_0\|}} \cancel{\|x_0\|}$$

$$= \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon$$

$\therefore \cdot$ is continuous operator.

Theorem (continuity and boundedness)

Let $T: D(T) \to Y$ be a linear operator where $D(T) \subseteq X$ and X, Y are normed space. Then:

(a) T is continuous if and only if T is bounded

(b) T is continuous at a single point, it is continuous.

proof: (a) when the linear operator $T = 0$ it is clearly that T is bounded and continuous

Let $T \neq 0$, and assume that T is bounded.

Let $x_0 \in D(T)$. for any $\epsilon > 0$. then $\delta = \frac{\epsilon}{\|T\|}$

$\because T \neq 0 \longrightarrow \|T\| \neq 0$

$\|x - x_0\| < \delta$ for every $x \in D(T)$.

now

$$\|T(x) - T(x_0)\| = \|T(x - x_0)\| \quad \text{since } T \text{ is linear operator}$$

$$\leq \|T\| \, \|x - x_0\| \quad \text{since } T \text{ is bounded operator.}$$

$$< \|T\| \, \delta$$

$$= \cancel{\|T\|} \frac{\epsilon}{\cancel{\|T\|}} = \epsilon$$

since $x_0 \in D(T)$ was arbitaray, this is shows that T is continuous.

(33) conversely

assume that T is continuous at an arbitrary point $x_0 \in D(T)$, for any $\epsilon > 0$, there is a $\delta > 0$ such that

$$\|T(x) - T(x_0)\| < \epsilon \quad \text{for all } x \in D(T)$$

whenever $\|x - x_0\| < \delta$

we now take any $0 \neq y \in D(T)$ which is an arbitrary

we set that

$$x = x_0 + \frac{\delta}{\|y\|} y$$

~~we see that~~

$$x - x_0 = \frac{\delta}{\|y\|} y$$

we see that
$$\|x - x_0\| = \Big\| \overset{\in K}{\frac{\delta}{\|y\|}} y \Big\|$$
$$= \frac{\delta}{\cancel{\|y\|}} \cancel{\|y\|}$$
since $y \neq 0$ implies $\|y\| = 0$
$$= \delta$$

hence $\|x - x_0\| = \delta$, we have that

$$\|T(x) - T(x_0)\| < \epsilon$$

$$\|T(x)-T(x_0)\| = \|T(x-x_0)\| \quad T \text{ is linear operator}$$
$$= \|T(\frac{\delta}{\|y\|} y)\|$$
$$= \|\frac{\delta}{\|y\|} T(y)\| \quad T \text{ is linear operator}$$
$$= \frac{\delta}{\|y\|} \|T(y)\| < \epsilon$$

$$\frac{\delta}{\|y\|} \|T(y)\| < \epsilon$$

$$\|T(y)\| < \frac{\epsilon}{\delta} \|y\| \qquad \forall\, y \in D(T)$$

set $\frac{\epsilon}{\delta} = c$

So $\|T(y)\| < c\,\|y\| \qquad \forall\, y \in D(T)$

$\therefore$ T is bounded operator.

(b) Let T is continuous at single point x_0 as we show in second part of (a) we get T is bounded on $D(T)$, now since T is bounded by using the first part we have T is continuous on $D(T)$.

Note:-

For linear operator $T: X \to Y$, where X, Y are normed space. when we want to prove continuity we need prove it at single point which always taken the zero element or we prove the boundedness which implies the continuity.

~~**Lemma** The scalar multiplication $\cdot : R \times X \to X$ where $(X, \|\cdot\|)$ is normed space we have $(R \times X, \|\cdot\|_2)$~~

~~$\|(\alpha, x)\|_2 = |\alpha| \, \|x\|$~~

~~**proof:** at first we to show that $\cdot$ is linear operator. Let $x, y \in R \times X$ such that $x = (\alpha, x_1)$, $y = (\beta, y_1)$~~

~~$\cdot(x+y) = \cdot((\alpha, x_1) + (\beta, y_1))$~~

~~$= \cdot((\alpha+\beta, x_1+y_1))$~~

~~$= (\alpha+\beta)\cdot(x_1$~~

Continuity, null space

Let T be ~~[struck out]~~ operator.
Then $T: X \to Y$, X, Y normed space.
T is continuous at x_0 if and only if
for $x_n \to x_0$ $(x_n \in X)$ implies
that $T(x_n) \to T(x_0)$

Proof: let T be continuous at x_0
now for $x_n \to x_0$, so for all $\epsilon > 0$
there exists $k \in N$ such that
$\|x_n - x\| < \epsilon \quad \forall n > k$
take $\epsilon = \delta$
when $\|x_n - x\| < \delta$ the condition satisfy
so $\|T(x_n) - T(x)\| < \epsilon \quad \forall n > k$
that is mean $T(x_n) \to T(x)$

Conversely let $x_n \to x_0$ implies $T(x_n) \to T(x_0)$
$\because x_n \to x_0$, so $\|x_n - x_0\| < \epsilon \quad \forall n > k$
take $\delta = \epsilon$, so $\|x_n - x_0\| < \delta$, so $\forall \epsilon > 0 \ \exists k_1 \in N$
$\|T(x_n) - T(x)\| < \epsilon \quad \forall n > k_1$
$\therefore T$ is continuous at x_0.

37 Lemma

Let T be a continuous operator at X. Then the null space $N(T)$ is closed.

proof: Let $x \in \overline{N(T)}$, by using theorem

* Let $(X, \|\cdot\|)$ be normed space. then $x \in \overline{M}$ if and only if there is a sequence (x_n) in M such that $x_n \longrightarrow x$.

we have (x_n) sequence in $N(T)$ such that $x_n \longrightarrow x$

$\because$ T is continuous by theorem

* $T: X \longrightarrow Y$ if T is continuous at x if and only if $x_n \longrightarrow x_0$ implies $T(x_n) \longrightarrow T(x_0)$

so $T(x_n) \longrightarrow T(x)$

$\because$ $(x_n) \in N(T)$ we have $T(x_n) = 0$ $\forall n \in N$

then from $T(x) = 0$ so we have that $x \in N(T)$.

Since $x \in \overline{N(T)}$ is arbitrary, so $N(T)$ is closed.

Example

Let $T: R^3 \longrightarrow R^3$ such that

$T(x) = T(x_1, x_2, x_3) = (x_2, x_3, 0)$

$\|x\| = |x_1| + |x_2| + |x_3|$

T is linear continuous operator and $N(T)$ is closed., find $R(T)$, $\dim(R(T))$.

Solution let $x, y \in R^3$, $x = (x_1, x_2, x_3)$
$y = (y_1, y_2, y_3)$

$$\begin{aligned}
T(\alpha x + \beta y) &= T(\alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3)) \\
&= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3) \\
&= (\alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3, 0) \\
&= \alpha(x_2, x_3, 0) + \beta(y_2, y_3, 0) \\
&= \alpha T(x) + \beta T(y)
\end{aligned}$$

T is linear operator.

$$\begin{aligned}
\|T(x)\| &= \|T(x_1, x_2, x_3)\| \\
&= \|(x_2, x_3, 0)\| \\
&= |x_2| + |x_3| + |0| \\
&= |x_2| + |x_3| \\
&\leq |x_1| + |x_2| + |x_3| \\
&= \|x\|
\end{aligned}$$

So T is bounded.

$\because$ T is linear bounded by using theorem

"Let $T: X \longrightarrow Y$ be linear operator and X, Y normed vector space, then T is bounded if and only if T is continuous on X."

(39)

$N(T) = \{ x \in X / T(x) = 0 \}$

$T(x_1, x_2, x_3) = (0, 0, 0)$

$(x_2, x_3, 0) = (0, 0, 0)$

$\rightarrow x_2 = 0$

$x_3 = 0$

$N(T) = \{ (x_1, 0, 0) / x_1 \in R \}$.

$\because$ T is continuous by using theorem

"LT be continuous at X, Then the nullspace N(T) is closed",

So N(T) is closed set.

$\dim(N(T)) = 1$ why?

$R(T) = \{ T(x) / x \in R^3 \}$

$= \{ (x, y, 0) / x \in R \}$

$\dim(R(T)) = 2$ why?

$\dim(X) = 3 = 1 + 2 = \dim(N(T)) + \dim(R(T))$