now set $\alpha = i$, then $\bar{\alpha} = -i$

$$i\langle T(x),y\rangle - i\langle T(y),x\rangle = 0 \quad \text{—} \; (2)$$

we get that

$$\langle T(x),y\rangle + \langle T(y),x\rangle = 0$$

$$i\langle T(x),y\rangle - i\langle T(y),x\rangle = 0$$

$$i\langle T(x),y\rangle + \cancel{i\langle T(y),x\rangle} = 0$$

$$i\langle T(x),y\rangle - \cancel{i\langle T(y),x\rangle} = 0 \qquad \text{بالجمع}$$

$$\frac{2i\langle T(x),y\rangle}{2i} = \frac{0}{2i}$$

$$\langle T(x),y\rangle = 0 \qquad \forall x \in X,\; y \in X$$

~~$\langle T(x), T(x)\rangle$ set $y = T(x)$~~

from the first part (1) we get that $T = 0$.

Note The second prove is true when X is vector space over complex number and it is not true when X be vector space over field of real number

Example Let $T: R^2 \to R^2$ such that

$$T(x_1, x_2) = (-x_2, x_1)$$

then T is bounded linear operator

$\alpha, \beta \in R$, $x = (x_1, x_2)$, $y = (y_1, y_2)$ $\quad x, y \in R^2$

$$T(\alpha x + \beta y) = T(\alpha(x_1, x_2) + \beta(y_1, y_2))$$
$$= T((\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2))$$
$$= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$
$$= (-(\alpha x_2 + \beta y_2), \alpha x_1 + \beta y_1)$$
$$= (-\alpha x_2, \alpha x_1) + (\beta y_2, \beta y_1)$$
$$= \alpha(-x_2, x_1) + \beta(-y_2, y_1)$$
$$= \alpha T(x) + \beta T(y)$$

$\therefore$ T is linear operator.

$$\|T(x)\|^2 = \langle T(x) / T(x) \rangle$$
$$= \langle (-x_2, x_1) / (-x_2, x_1) \rangle$$
$$= -x_2 x_2 + x_1 x_1$$
$$= x_1^2 + x_2^2 = \langle (x_1, x_2) / (x_1, x_2) \rangle = \|x\|^2$$

$$\|T(x)\| = \sqrt{x_1^2 + x_2^2}$$

$$\|x\|^2 = \langle (x_1, x_2) / (x_1, x_2) \rangle$$
$$= x_1^2 + x_2^2$$

$$\|x\| = \sqrt{x_1^2 + x_2^2}$$

~~$\|T(x)\| \leq \|x\|$~~ —

$$\|T\| = \sup_{0 \neq x \in X} \left\{ \frac{\|T(x)\|}{\|x\|} = \sup_{0 \neq x \in X} \frac{\|x\|}{\|x\|} \right\} = 1$$

but not satisfies (2) in zero operator Lemma

Since $x = \langle x_1, x_2 \rangle$

$$\langle T(x) / x \rangle = \langle (-x_2, x_1) / (x_1, x_2) \rangle$$

$\forall x \in X$

$$= -x_2 x_1 + x_1 x_2 = 0$$

$\in \mathbb{R}$

by $T(x) \neq 0$

and so $T \neq 0$

since $X = \mathbb{R}^2$ is a real vector space not complex.

<u>Theorem</u> (properties of Hilbert-adjoint operator)

Let H_1, H_2 be Hilbert space

$T : H_1 \longrightarrow H_2$

$S : H_1 \longrightarrow H_2$

T, S are bounded linear operators and

$\alpha \in K$. Then we have

(1) $\langle T^* y, x \rangle = \langle y / T(x) \rangle$

(2) $(S+T)^* = S^* + T^*$

(3) $(\alpha T)^* = \bar{\alpha} T^*$

(4) $(T^*)^* = T$

(5) $\|T^*T\| = \|TT^*\| = \|T\|^2$

(6) $T^*T = 0 \longleftrightarrow T = 0$

(7) $(ST)^* = T^* S^*$

proof: (1) Let $x \in H_1$, $y \in H_2$.

$\langle T^*(y) / x \rangle = \overline{\langle x / T^*(y) \rangle}$

$= \overline{\langle T(x) / y \rangle}$

$= \langle y / T(x) \rangle$

$\therefore \langle T^*(y) / x \rangle = \langle y / T(x) \rangle$

(2) $\langle x / (S+T)^*(y) \rangle = \langle (S+T)(x) / y \rangle$ by definition of adjoint operator

$= \langle S(x) + T(x) / y \rangle$

$= \langle S(x) / y \rangle + \langle T(x) / y \rangle$

$= \langle x / S^*(y) \rangle + \langle x / T^*(y) \rangle$

$= \langle x / S^*(y) + T^*(y) \rangle$

hence $(S+T)^*(y) = S^*(y) + T^*(y) \quad \forall y \in H_2$

$= (S^* + T^*)(y)$

$\therefore (S+T)^* = S^* + T^*$

③ we to show that $T^*(\alpha x) = \alpha T^*(x)$

$\cancel{\langle (\alpha T)^* / x \rangle}$

$\langle (\alpha T)^*(y) / x \rangle = \langle y / (\alpha T)(x) \rangle$

$= \langle y / \alpha T(x) \rangle$

$= \bar{\alpha} \langle y / T(x) \rangle$

$= \bar{\alpha} \langle T^*(y) / x \rangle$

$= \langle \bar{\alpha} T^*(y) / x \rangle$

So $(\alpha T)^*(y) = \bar{\alpha} T^*(y) \quad \forall y \in H_2$

$\therefore (\alpha T)^* = \bar{\alpha} T^*$

④ $(T^*)^*$ is written as T^{**} which is equal T

$\langle (T^*)^* x / y \rangle = \langle x / T^*(y) \rangle$

~~by definition of T^*~~

$= \langle T(x) / y \rangle$

$(T^*)^*(x) = T(x) \quad \forall x \in H_1$

$\therefore (T^*)^* = T$

⑤ we see that $T^* T : H_1 \longrightarrow H_1$

$T T^* : H_2 \longrightarrow H_2$

$\| T(x) \|^2 = \langle T(x) / T(x) \rangle$

$= \langle T^* T(x) / x \rangle$ by Schwarz inquality

$\leq \| T^* T(x) \| \cdot \| x \|$

$\leq \| T^* T \| \, \| x \|^2$

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$$\|T\|^2 = \sup_{0\neq x\in H_1} \left\{\frac{\|T(x)\|^2}{\|x\|^2}\right\}$$

$$\le \sup_{0\neq x\in H_1}\left\{\frac{\|T^*T\|\,\cancel{\|x\|^2}}{\cancel{\|x\|^2}}\right\}$$

$$= \|T^*T\|$$

$$\therefore \|T\|^2 \le \|T^*T\| \le \|T^*\|\,\|T\|$$

$\because \|T^*\| = \|T\|$ we get

$$\le \|T\|^2$$

So $\|T\|^2 = \|T^*T\|$

~~Similarly~~ we can prove $\|TT^*\| = \|T\|^2$

by replacing T by T^*

$$\|(T^*)^*T^*\| = \|T^*\|^2$$

$$\|TT^*\| = \|T\|^2$$

So $\|T^*T\| = \|TT^*\| = \|T\|^2$

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(6) Let $\cancel{T^*T \to 0}$

$$T^*T = 0 \longrightarrow 0 = \|T^*T\| = \|T\|^2 \text{ by (5)}$$

$$\Rightarrow \|T\|^2 = 0$$

$$\Leftrightarrow \|T\| = 0 \quad \text{so} \quad T = 0$$

conversely if $T = 0$

$$T^*T = 0$$

(7) $(ST)^* = T^* S^*$

$$\langle x / (ST)^* y \rangle = \langle ST(x) / y \rangle$$

$$= \langle T(x) / S^*(y) \rangle$$

$$= \langle x / T^* S^*(y) \rangle$$

$$\langle x / (ST)^* y - T^* S^*(y) \rangle = 0$$

$$\langle x / ((ST)^* + T^* S^*)(y) \rangle = 0$$

by zero operator lemma

$$(ST)^* - T^* S^* = 0$$

$$(ST)^* = T^* S^*$$
