$[1] = \{x \in \mathbb{Z}, x R 1\}$

$= \{x : \exists k \in \mathbb{Z}, x-1 = 3k\}$

$= \{x : \exists k \in \mathbb{Z}, x = 3k+1\}$

$= \{1, 4, -2, 7, \dots\}$

$[4] = \{x \in \mathbb{Z}, x R 4\}$

$= \{x : \exists k \in \mathbb{Z}, x-4 = 3k\}$

$= \{x : \exists k \in \mathbb{Z}, x = 3k+4\}$

$= \{4, 7, 1, \dots\}$

$[-1] = \{x : \exists k \in \mathbb{Z}, \underbrace{x-(-1)}_{x+1} = 3k\}$

$= \{x : \exists k \in \mathbb{Z}, x = 3k-1\}$

$= \{-1, 2, -4, 7, \dots\}$

Ex.:- Let $R = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : a-b = \text{even}\}$. Find $[1], [3]$

Sol.:- $R = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : a-b = 2k, k \in \mathbb{Z}\}$

$= \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : a = 2k+b, k \in \mathbb{Z}\}$

اولاً نختبر R يجب ان تكون علاقة تكافؤ R is equivalence relation.

$[1] = \{a : \exists k \in \mathbb{Z}, a-1 = 2k\}$

$= \{a : \exists k \in \mathbb{Z}, a = 2k+1\}$

$= \{1, 3, -1, 5, \dots\}$

$[3] = \{a : \exists k \in \mathbb{Z}, a = 2k+3\}$

$= \{3, 5, 1, 7, -1, \dots\}$

$\therefore [1] = [3]$.

Theorem:- Let R be an equivalence relation in A. (24)

Then 1- $a \in [a]$, $\forall a \in R$.

2- if $b \in [a]$ and $a \in [b]$, then $[a] = [b]$.

3- $[a] = [b]$ iff $(a,b) \in R$.

4- if $[a] \cap [b] \neq \emptyset$, then $[a] = [b]$.

Proof(1):- $[a] = \{x \in A : (x,a) \in R\}$

$\because$ R is equivalence relation in A. Then R is reflexive

$\Rightarrow \forall a \in A$, $(a,a) \in R \Rightarrow a \in [a]$ (by def. of $[a]$).

Proof(2):- let $x \in [a] \Rightarrow (x,a) \in R$

$\because b \in [a] \Rightarrow (b,a) \in R$

since R is symmetric $\Rightarrow (a,b) \in R$

$\because (x,a) \in R$ and $(a,b) \in R$

$\Rightarrow (x,b) \in R$ (R is transitive)

$\Rightarrow x \in [b]$

$\therefore [a] \subseteq [b]$ ---- ①

Now, let $y \in [b] \Rightarrow (y,b) \in R$

$a \in [b] \Rightarrow (a,b) \in R$

$\because$ R is symmetric $\Rightarrow (b,a) \in R$

$\therefore (y,b) \in R$ and $(b,a) \in R$

$\Rightarrow (y,a) \in R$ (R is transitive)

$\Rightarrow y \in [a]$

$\therefore [b] \subseteq [a]$ --- ②

From ① and ② we get $[a] = [b]$.

(3) suppose that $[a]=[b]$

$\because a\in[a]$ from (1)

$\because [a]=[b] \Rightarrow a\in[b] \Rightarrow (a,b)\in R$

$\Leftarrow$ assume that $(a,b)\in R$, to prove $[a]=[b]$

let $x\in[a] \Rightarrow (x,a)\in R$

$\because (a,b)\in R \Rightarrow (x,b)\in R$ (since R is tra.)

$\Rightarrow x\in[b] \Longrightarrow [a]\subseteq[b]$ ---①

Now, let $y\in[b] \Rightarrow (y,b)\in R$ $\therefore (a,b)\in R$

$\Rightarrow (b,a)\in R$ since (R is symm.)

$\Rightarrow (y,b)$ and $(b,a)\in R$

$\Rightarrow (y,a)\in R$ since (R is tra.) $\Rightarrow y\in[a]$

$\Rightarrow [b]\subseteq[a]$ ---②

from ① and ② we get $[a]=[b]$.

proof(4):- $\because [a]\cap[b]\neq\emptyset$

$\Rightarrow z\in[a]\cap[b]$

$\Rightarrow \exists z\in[a]$ and $z\in[b]$

$\Rightarrow (z,a)\in R$ and $(z,b)\in R$

$\Rightarrow (a,z)\in R$ and $(z,b)\in R$ (since R is sym.)

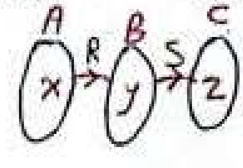
$\Rightarrow (a,b)\in R$ since (R is tra.)

from (3) we get

$[a]=[b]$.

Def.:- If R is a relation from A into B and (25)
S is a relation from B into C, then تركيب العلاقات

$$S \circ R = \{ (x,z) \in A \times C ,\ \exists\, y \in B \text{ s.t } (x,y) \in R \text{ and } (y,z) \in S \}$$



Note:- $R \circ S \neq S \circ R$

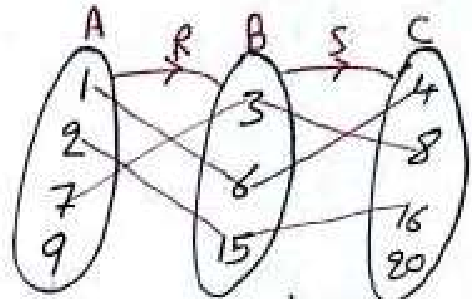
Ex.:- let $A = \{1,2,7,9\}$, $B = \{3,6,15\}$, $C = \{4,8,16,20\}$

$R = \{ (1,6), (2,15), (7,3) \}$

$S = \{ (3,8), (15,16), (6,4) \}$

$S \circ R = \{ (1,4), (2,16), (7,8) \}$

$R \circ S = \phi$



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Theorem:- Let R be a relation defined on A. Then

$$R \circ I_A = I_A \circ R = R .$$

Proof:-

iden. relation

let $(x,z) \in R \circ I_A$ $\quad (x,y) \in I_A \wedge (y,z) \in R$

$\Rightarrow \exists\, y \in A$ s.t $(x,y) \in I_A$ and $(y,z) \in R$

$\because (x,y) \in I_A \Rightarrow x = y \Rightarrow (x,x) \in I_A$ and $(y,z) \in R$

$\Rightarrow (x,z) \in R \Rightarrow R \circ I_A \subseteq R$ --- (1)

Now, let $(x,z) \in R \Rightarrow (x,x) \in I_A$ and $(x,z) \in R$

$\Rightarrow (x,z) \in R \circ I_A$

$\Rightarrow R \subseteq R \circ I_A$ --- (2)

From (1) and (2) we get $R \circ I_A = R$.

Th. let R, S and T are relations defined on the set A. Then

① $(T \circ S) \circ R = T \circ (S \circ R)$.

② $(S \cup T) \circ R = (S \circ R) \cup (T \circ R)$.

③ $(S \cap T) \circ R = (S \circ R) \cap (T \circ R)$.

④ If $R \subseteq S$, then $T \circ R \subseteq T \circ S$, $(R \circ T \subseteq S \circ T)$.

⑤ $(S \circ R) \cap T = \emptyset$ (then) iff $(T \circ R^{-1}) \cap S = \emptyset$.

⑥ $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$ and $(R \circ S)^{-1} = S^{-1} \circ R^{-1}$.

① $(T \circ S) \circ R = T \circ (S \circ R)$

proof:- let $(x,z) \in (T \circ S) \circ R$

$\Leftrightarrow \exists y \in A$ s.t $(x,y) \in R$ and $(y,z) \in (T \circ S)$

$\Leftrightarrow \exists y \in A$ s.t $(x,y) \in R$ and $\exists l \in A$ s.t $(y,l) \in S$ and $(l,z) \in T$

$\Leftrightarrow \exists y \in A$ and $l \in A$ s.t $(x,y) \in R$ and $(y,l) \in S$ and $(l,z) \in T$

$\Leftrightarrow \exists l \in A$ s.t $(x,l) \in (S \circ R)$ and $(l,z) \in T$

$\Leftrightarrow (x,z) \in T \circ (S \circ R)$

$\Leftrightarrow (T \circ S) \circ R \subseteq T \circ (S \circ R)$ ---- ①

وبعكس الخطوات نحصل على

$T \circ (S \circ R) \subseteq (T \circ S) \circ R$ ---- ②

from ① and ② we get

$(T \circ S) \circ R = T \circ (S \circ R)$.