Stokes's Theorem
نــظـــريــــة ستوكس

Theorem: Let S be apiece wise smooth oriented surface that is bounded by a simple, closed apiece wise smooth curve C with positive orientation. If the components(مركبات) of the vector field $F(x,y,z) = f(x,y,z)i + g(x,y,z)j + h(x,y,z)k$ are continuous and have continuous first partial derivatives on some open set containing S , then

$$\oint_C F.dR = \iint_S Curl\, F.n.ds$$

Where $R = xi + yj + zk$

$$dR = dxi + dyj + dzk$$

Example 1: Verify Stokes's theorem for field $F(x,y) = yi - xj$ on the hemisphere $z = \sqrt{a^2 - x^2 - y^2}$

Solution:

$$\oint_C F.dR = \iint_S Curl\, F.n.ds$$

$$x^2 + y^2 + z^2 - a^2 = 0$$

$$\iint_S Curl\, F.n.ds$$

$$Curl\, F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = (0-0)i - (0-0)j + (-1-1)k = -2k$$

$$\therefore\ Curl\, F = -2k$$

$$n = \frac{\nabla F}{\|\nabla F\|} = \frac{2xi + 2yj + 2zk}{\sqrt{4x^2 + 4y^2 + 4z^2}} \quad , ds = \frac{\sqrt{4x^2 + 4y^2 + 4z^2}}{2z}\, dA$$

$$= \iint_S -2\,k\,.\frac{2xi+2yj+2zk}{\sqrt{4x^2+4y^2+4z^2}}.\frac{\sqrt{4x^2+4y^2+4z^2}}{2z}\ dA \Rightarrow$$

$$= \iint_S \frac{-4z}{2z}\ dA \Rightarrow W = -2\iint_S\ dA \Rightarrow$$

$$= -2\int_0^{2\pi}\int_0^a r dr\, d\theta \quad = -2\int_0^{2\pi}\ \frac{r^2}{2}\Big]_0^a\ d\theta \quad = -\int_0^{2\pi} a^2\, d\theta = -2a^2\pi.$$

$$\oint_C\ F.dR =$$

$$\oint_C\ (yi-xj)\,.(dxi+dyj+dzk) = \oint_C\ (ydx-xdy) = \iint_R (-1-1)dA$$

$$= -2\iint_R\ dA = -2\int_0^{2\pi}\int_0^a r\,dr\,d\theta = -2\int_0^{2\pi}\frac{r^2}{2}\Big]_0^a d\theta = -2a^2\pi.$$

Example 2: Verify Stokes's theorem for the field $F = xi+yj+zk$ on the surface $z = x^2+y^2$, $z \le 1$

Solution:

$$\oint_C\ F.dR = \iint_S Curl\,F.n.ds$$

$$z - x^2 - y^2 = 0$$

$$\iint_S Curl\,F.n.ds$$

$$Curl\,F = \nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = (0-0)i-(0-0)j+(0-0)k = 0$$

$\therefore\ Curl\,F = 0$

$$n = \frac{\nabla F}{\|\nabla F\|} = \frac{2xi+2yj+k}{\sqrt{4x^2+4y^2+1}}\quad , ds = \frac{\sqrt{4x^2+4y^2+1}}{1}\ dA$$

$$= \iint_S 0 \,.\frac{2xi + 2yj + 2zk}{\sqrt{4x^2 + 4y^2 + 1}}.\frac{\sqrt{4x^2 + 4y^2 + 1}}{1}\, dA = 0$$

$$\oint_C F.dR =$$

$$\oint_C (xi + yj + zk).(dxi + dyj + dzk) = \oint_C (xdx + ydy + zdz)$$

$$\oint_C (\cos\theta \;\, d(\cos\theta) + sin\theta \, d(sin\theta) + 1d(1)$$

$$\int_0^{2\pi} -cos\theta \, sin\theta + sin\theta \, cos\theta + 0 = 0$$