⋮

$x_n = f(x_{n-1}) = f^n(x_0)$

Then

$d(x_1, x_2) = d\big(f(x_0), f(x_1)\big) \le cd(x_0, x_1)$

$\Rightarrow\ d(x_2, x_3) = d\big(f(x_1), f(x_2)\big) \le cd(x_1, x_2) \le c^2 d(x_0, x_1)$

⋮

$\Rightarrow\ d(x_n, x_{n+1}) \le c^n d(x_0, x_1)$

We will prove that $< x_n >$ is Cauchy sequence

Now, for any $m, n \in N$ with $m > n$

$$
\begin{aligned}
d(x_n, x_m) &\le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&\le c^n d(x_0, x_1) + c^{n+1} d(x_0, x_1) + \cdots + c^{m-1} d(x_0, x_1) \\
&\le (c^n + c^{n+1} + c^{n+2} + \cdots + c^{m-1}) d(x_1, x_0) \\
&\le c^n (1 + c + c^2 + c^3 + \cdots + c^{m-n-1}) d(x_1, x_0) \\
&\le c^n \left(\textstyle\sum_{k=1}^{n-m} c^{k-1}\right) d(x_1, x_0) \\
&\le c^n \left(\textstyle\sum_{k=1}^{\infty} c^{k-1}\right) d(x_1, x_0) \\
&\le \left(\frac{c^n}{1-c}\right) d(x_1, x_0)
\end{aligned}
$$

Since $0 \le c < 1$

$\Rightarrow\ < x_n > \to 0$ as $n \to \infty$

$\Rightarrow\ < x_n >$ is Cauchy sequence

Since X is complete

$\Rightarrow\ < x_n > \to x\ ,\ x \in X$

$\Rightarrow\ f(x) = f\,(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$

To prove the uniqueness, if x and y are two fixed point, then

$0 \le d(x, y) = d\big(f(x), f(y)\big) \le cd(x, y)$

Since $c < 1$, we have $d(x, y) = 0\ \Rightarrow\ x = y$

Hence, f has a unique fixed point.

Compactness in Metric Spaces التراص في الفضاءات المترية

Definition (4.15): Let E be a subset in a metric space (X, d)

(i) We say that the family $\{G_\alpha\}_{\alpha\in\Lambda}$ is a **cover** (غطاء) for E iff $E \subset \bigcup_{\alpha\in\Lambda} G_\alpha$.

(ii) We say that the subfamily of $\{G_\alpha\}_{\alpha\in\Lambda}$ (say $\{G_{\alpha_i}\}_{\alpha_i\in\Lambda}$ is a **subcover** (غطاء جزئي) of E) iff

$E \subset \bigcup_{\alpha_i\in\Lambda} G_{\alpha_i}$.

(iii) We say that the family $\{G_\alpha\}_{\alpha\in\Lambda}$ is an **open cover** (غطاء مفتوح) of E iff $E \subset \bigcup_{\alpha\in\Lambda} G_\alpha$ and G_α is an open set, $\forall\ \alpha \in \Lambda$.

(iv) We say that $\{G_{\alpha_i}\}_{i=1}^n$ is an **open finite subcover** (غطاء جزئي منتهي مفتوح) of E iff $E \subset \bigcup_{i=1}^n G_{\alpha_i}$, where G_{α_i} is an open set.

Definition (4.16): (Compact Set) المجموعة المتراصة

If E is a subset in (X, d). We say that E is **compact** iff for every open cover $\{G_\alpha\}_{\alpha\in\Lambda}$ of E, $\exists$ a finite subcover of E (say $\{G_{\alpha_i}\}_{i=1}^n$).

<u>In other word</u>: (E compact) $\Leftrightarrow \left[\left(E \subset \bigcup_{\alpha\in\Lambda} G_\alpha\right) \Rightarrow \left(E \subset \bigcup_{i=1}^n G_{\alpha_i}\right)\right]$

Proposition (4.3): Every finite subset in a metric space (X, d) is compact.

Proof: Let $E = \{x_1, x_2, x_3, \dots, x_n\}$ be a finite v subset in (X, d)

Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be an open cover of E

$\Rightarrow E \subset \bigcup_{\alpha\in\Lambda} G_\alpha \Rightarrow \exists\ G_{\alpha_i}, x_i \in G_{\alpha_i}, i = 1,2, \dots, n$

$\Rightarrow x_1 \in G_{\alpha_1}, x_2 \in G_{\alpha_2}, \dots\ x_n \in G_{\alpha_n}$

$\Rightarrow \{x_1\} \subset G_{\alpha_1}, \{x_2\} \subset G_{\alpha_2}, \dots\ \{x_n\} \subset G_{\alpha_n}$

$\Rightarrow \bigcup_{i=1}^n \{x_i\} \subset \bigcup_{i=1}^n G_{\alpha_i}$

$\Rightarrow E \subset \bigcup_{i=1}^n G_{\alpha_i}$

$\Rightarrow \{G_{\alpha_i}\}_{i=1}^n$ is a finite subcover of E.

$\Rightarrow E$ is compact.

Example (4.18): Determine whether $E = (0,1)$ is compact or not in (R, d).

Proof: Let $\{G_n\}_{n\in N}$ such that $G_n = (\frac{1}{n+1}, 1)$ be an open cover of E.

$\Rightarrow E \subset \bigcup_{\alpha\in\Lambda} (\frac{1}{n+1}, 1)$

Let (a_1, b_1), (a_2, b_2), … , (a_m, b_m) be a finite subfamily of $\left\{\left(\frac{1}{n+1}, 1\right)\right\}_{n\in N}$

Put $\varepsilon = min(a_1, a_2, \dots, a_m)$

$\Rightarrow \bigcup_{i=1}^{m}(a_i, b_i) = (\varepsilon, b_m)$

Let $b_m = 1$

$\Rightarrow \bigcup_{i=1}^{m}(a_i, b_i) = (\varepsilon, 1)$

Now $(0,1) \not\subset (\varepsilon, 1)$, for any $\varepsilon > 0$

$\Rightarrow (0,1) \not\subset \bigcup_{i=1}^{m}(a_i, b_i)$

There is no finite subcover for E

$\Rightarrow E$ is not compact.

Theorem (4.6): The closed interval $[a, b]$ is compact in (R, d).

Example (4.19): The set $E = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, 0\right\}$ is a compact set in (R, d)

Proof: Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be an open cover of E

$\Rightarrow E \subset \bigcup_{\alpha\in\Lambda} G_\alpha$

Let $0 \in G_{\alpha_0}$, $\alpha_0 \in \Lambda$

$\because$ G_{α_0} is an open set, then $\exists$ an open interval $(-r, r) \subseteq G_{\alpha_0}$

$\Rightarrow 0 \in (-r, r)$

By Archimedean property, $\exists$ $k \in N$ s.t. $1 < kr \Rightarrow \frac{1}{k} < r$

$\Rightarrow \frac{1}{n} < \frac{1}{k} < r$, $\forall$ $n > k$

$\Rightarrow \frac{1}{n} \in (-r, r)$, $\forall\, n > k$

$\Rightarrow \frac{1}{n} \in G_{\alpha_0}$, $\forall\, n > k$

Let G_{α_i} be an open set in the cover $\{G_\alpha\}_{\alpha\in\Lambda}$ s.t. $\frac{1}{i} \in G_{\alpha_i}$

Now, $\{G_{\alpha_1}, G_{\alpha_2}, \dots, G_{\alpha_k}\}$ is a finite open cover and contains E .

$\Rightarrow$ E is compact.

Remark (4.2): If X is compact, then we say that (X, d) is a compact metric space.

Theorem (4.7): Every closed subset of a compact metric space (X, d) is compact.

Proof: Let $E \subset X$ be a closed set

Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be an open cover of E

$\Rightarrow$ $E \subset \bigcup_{\alpha\in\Lambda} G_\alpha$

Since E closed

$\Rightarrow$ E^c open and

Since $E \cup E^c = X$

$\Rightarrow$ $E \cup E^c \subset (\bigcup_{\alpha\in\Lambda} G_\alpha) \cup E^c$

$\Rightarrow$ $X \subset \bigcup_{\alpha\in\Lambda} (G_\alpha \cup E^c)$

$\Rightarrow$ $\{G_\alpha \cup E^c)\}_{\alpha\in\Lambda}$ is an open cover of X

Since X is a compact space

$\Rightarrow$ $X \subset \bigcup_{i=1}^{n} G_{\alpha_i} \cup E^c$

We have $E \subset X$

$\Rightarrow$ $E \subset \bigcup_{i=1}^{n} G_{\alpha_i} \cup E^c$

$\Rightarrow$ $E \subset \bigcup_{i=1}^{n} G_{\alpha_i}$ (because $E \cap E^c = \emptyset$)

$\Rightarrow$ E is compact.
