Applied of Double Integration in Calculate Volumes
تطبيقات التكاملات الثنائية في حساب المساحات والحجوم

Definition: If f is integrals over a plane region R and $f(x,y) \geq 0$ for all (x,y) in R, then the volume of the solid region that line above R and below the graph of f is defined by:

$$V = \iint_R Z\, dA = \iint_R Z\, dxdy = \iint_R Z\, dydx$$

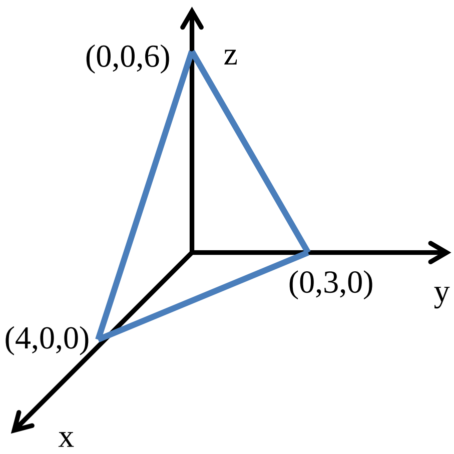
Example 1: Let the region R lie (تقع) in the first quadrant (الربع الاول) of the xy-plane, and bounded by $y = 0$, $y^2 = x$, and $x + 2y = 3$. Let the surface be the plane $3x + 4y + 2z = 12$. (1)- Sketch (ارسم) the solid and (2)- find the volume for this particular instance of the foregoing discussion

Solution:

Let $x = y = 0 \quad \Rightarrow \quad 2z = 12 \quad \Rightarrow z = 6$

$x = z = 0 \quad \Rightarrow \quad 4y = 12 \quad \Rightarrow y = 3$

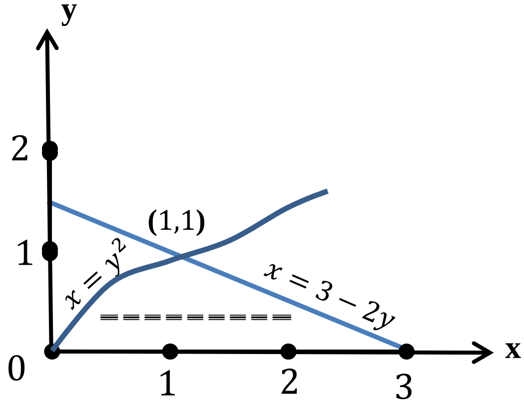
$y = z = 0 \quad \Rightarrow \quad 3y = 12 \quad \Rightarrow x = 4$



$y^2 = x \Rightarrow y = \pm\sqrt{x}$

x	y
0	0
1	1
2	1.2
3	1.7

$x + 2y = 3 \Rightarrow y = \dfrac{3 - x}{2}$



x	y
0	3/2
1	1
2	1/2
3	0

$$V = \iint_R Z\,dA = \iint_R \frac{1}{2}[12 - 3x - 4y]\,dA$$

$$\frac{1}{2}\int_{0}^{1}\int_{y^2}^{3-2y}[12 - 3x - 4y]\,\boldsymbol{dx\,dy} = \frac{1}{2}\int_0^1 \left[12x - \frac{3}{2}x^2 - 4yx\right]_{y^2}^{3-2y} dy$$

$$= \frac{1}{2}\int_0^1 \left[12(3-2y) - \frac{3}{2}(3-2y)^2 - 4y(3-2y) - \left[12y^2 - \frac{3}{2}y^4 - 4y^3\right]\right] dy$$

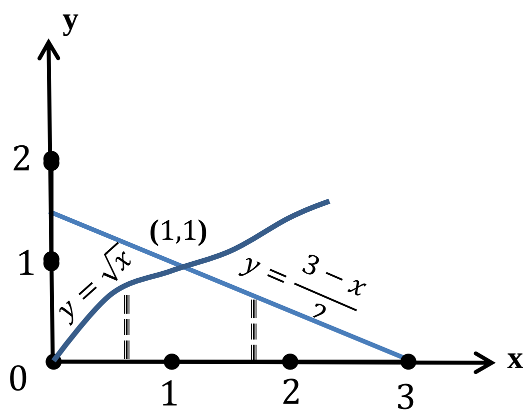
$$= \frac{1}{2}\int_0^1 \left[36 - 24y - \frac{3}{2}(9 - 12y + 4y^2) - 12y + 8y^2 - 12y^2 + \frac{3}{2}y^4 + 4y^3\right] dy$$

$$= \frac{1}{2}\int_0^1 \left[\frac{45}{2} - 18y - 10y^2 + 4y^3 + \frac{3}{2}y^4\right] dy$$

$$= \frac{1}{2}\left[\frac{45}{2}y - 9y^2 - \frac{10}{3}y^3 + y^4 + \frac{3}{10}y^5\right]_0^1 = \frac{86}{15}$$

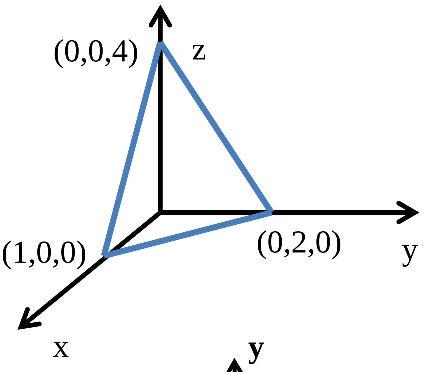
Reversing:

$$\int_0^1\int_0^{\sqrt{x}} \boldsymbol{z\,dydx} + \int_1^3\int_0^{\frac{3-x}{2}} \boldsymbol{z\,dydx} =$$



Example 2: Use a double integral to find the volume of the region R bounded by the coordinated planes and the plane $z = 4 - 4x - 2y$

Solution:



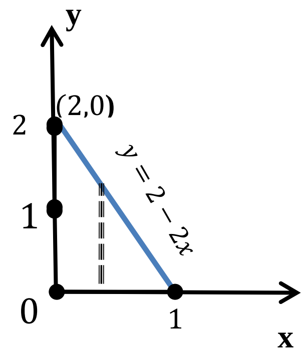
Let $x = y = 0 \quad \Rightarrow \quad z = 4 \quad \Rightarrow (0,0,4)$

$x = z = 0 \quad \Rightarrow \quad 2y = 4 \quad \Rightarrow (0,2,0)$

$y = z = 0 \quad \Rightarrow \quad y = 1 \quad \Rightarrow (1,0,0)$

$$V = \iint_R z\, dA = \iint_R 4 - 4x - 2y\, dA$$

Let $z = 0 \quad \Rightarrow \quad 4 - 4x - 2y = 0 \Rightarrow y = 2 - 2x$



X	y
0	2
1/2	1
1	0

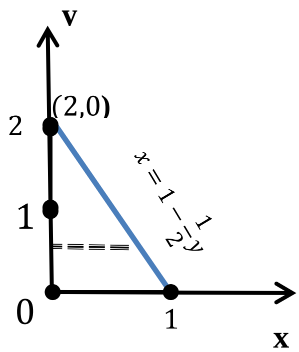
$$\int_0^1 \int_0^{2-2x} [4 - 4x - 2y]\,\boldsymbol{dy\, dx} = \int_0^1 [4y - 4xy - y^2]_{\mathbf{0}}^{2-2x}\, dx$$

$$= \int_0^1 \; 4(2-2x) - 4x\,(2-2x) - (2-2x)^2\, dx$$

$$= \int_0^1 8 - 8x - 8x + 8x^2 - (4 - 8x + 4x^2)dx$$

$$\int_0^1 [4x^2 - 8x + 4]dx = \left[\frac{4}{3}x^3 - 4x^2 + 4x\right]_0^1 = \frac{4}{3} - 4 + 4 = \frac{4}{3}$$

Reversing:



$$\int_0^2 \int_0^{1-\frac{1}{2}y} [4 - 4x - 2y]\,\boldsymbol{dx\, dy} = \int_0^2 [4x - 2x^2 - 2xy]_{\mathbf{0}}^{1-\frac{1}{2}y}\, dx$$

$$= \int_0^2 \; 4\left(1 - \frac{1}{2}y\right) - 2\,(1 - \frac{1}{2}y)^2 - 2y\left(1 - \frac{1}{2}y\right)\, dx$$

$$= \int_0^2 \left[4 - 2y - 2\left(1 - y + \frac{1}{4}y^2\right) - 2y + y^2\right] dx$$

$$= \int_0^2 \left[4 - 2y - 2 + 2y - \frac{1}{2}y^2 - 2y + y^2\right] dx = \int_0^2 \left[\frac{1}{2}y^2 - 2y + 2\right] dx$$

$$\left[\frac{1}{6}y^3 - y^2 + 2y\right]_0^2 = \frac{8}{6} - 4 + 4 = \frac{4}{3}$$