10^{th} Lecture

Theorem (3.2): Let (X,τ) be a topological space. If $X = A/B$ (i.e. X is separable set). Then A, B are both open and closed distinct subsets of X.

Proof:

We have $X = A/B$

$\Rightarrow\ A, B \neq \emptyset$, $A \cap B = \emptyset$, Also

$(A \cap \bar{B}) \cup (B \cap \bar{A}) = \emptyset$

$\Rightarrow\ A \cap \bar{B} = \emptyset\ \wedge\ B \cap \bar{A} = \emptyset$

If $A \cap \bar{B} = \emptyset$

$\Rightarrow\ A \cap (B \cup d(B)) = \emptyset$

$\Rightarrow\ (A \cap B) \cup (A \cup d(B)) = \emptyset$

But $A \cap d(B) = \emptyset$

$\Rightarrow$ No point of A is a limit point of B

$\Rightarrow$ limit points of B all in B

$\Rightarrow\ d(B) \subset B\ \Rightarrow\ B$ is closed

$\Rightarrow\ B^c$ is open $\Rightarrow\ A$ is open

Also if $B \cap \bar{A} = \emptyset$

$\Rightarrow$ all points of A are in A

$\Rightarrow\ A$ is closed

$\Rightarrow\ A^c$ is open

$\Rightarrow\ B$ open

Theorem (3.3): If C is connected in (X,τ) and $X = A/B$. Then $C \subset A$ or $C \subset B$.

Theorem (3.4): If C is connected in (X,τ) and $C \subset E \subset \bar{C}$. Then E is connected in (X,τ).

Proof:

Assume (if possible) E is not connected

$\Rightarrow\ E = A/B,\ A,B \neq \emptyset$

Now C is connected in $E = A/B$

$\Rightarrow\ C \subset A$ or $C \subset B$

If $C \subset A\ \Rightarrow\ \bar{C} \subset \bar{A}\ \Rightarrow\ \bar{C} \cap B \subset \bar{A} \cap B = \emptyset$

$\Rightarrow\ \bar{C} \cap B = \emptyset$ ………………. (1)

$B \subset E\ \wedge\ E \subset \bar{C}\ \Rightarrow\ B \subset \bar{C}$

$\Rightarrow\ \bar{C} \cap B = B$ ………………. (2)

From (1) and (2) we get

$B = \emptyset\ \Rightarrow$ Contradiction

$\Rightarrow\ E$ is connected.

Theorem (3.5): A topological space (X,τ) is connected iff the only open and closed sets in X are $\emptyset, X$.

Proof:

Suppose that (X,τ) is connected.

Assume that $\exists\ A \neq \emptyset$ and $A \subset X$ is both open and closed

$\Rightarrow\ B = A^c$ is both open and closed

We have $A,B \neq \emptyset$, $A \cap B = \emptyset$, $A \cup B = X$

Also $(\bar{A} \cap B) \cup (A \cap \bar{B}) = \emptyset$

$\Rightarrow\ X = A/B\ \Rightarrow$ Contradiction

Hence the only open closed sets in X are $\emptyset, X$

Conversely: Suppose that $\emptyset, X$ are the only open closed in X

We need to prove that X is connected

Assume that X is not connected

$\Rightarrow$ $X = A/B$

$\Rightarrow$ both A and B are open and closed

$\Rightarrow$ Contradiction

$\therefore$ (X, τ) is connected.
