Fourier Transforms

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5.1. INTRODUCTION

Fourier* transforms play an important part in the theory of many branches of science. The use of Fourier transforms is indispensible in solving the problems in the field of mathematics, physics and engineering. Many types of integrals and differential equations can be solved by using Fourier transforms.

5.2. FOURIER'S INTEGRAL THEOREM

Statement. *Let f be a real valued function of the real variable x such that f(x) and f′(x) are piecewise continuous in every finite interval and the integral of |f(x)| from – ∞ to ∞ exists. Then f(x) can be expressed as*

$$f(x) = \int_0^\infty (A(s) \cos sx + B(s) \sin sx)\, ds, \text{ where}$$

$$A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \cos sv\, dv \quad \text{and} \quad B(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \sin sv\, dv.$$

At a point where f(x) is discontinuous the value of the integral on the right side equals the average of left hand limit and right hand limit of f(x) at that point.

Note. The proof of this theorem is beyond the scope of this book.

The integral $\int_0^\infty (A(s) \cos sx + B(s) \sin sx)\, ds$ is called the **Fourier integral expression** of the function $f(x)$.

Theorem. *Prove that the Fourier integral expression of a function f(x) is same as the integral* $\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(v)\, e^{is(x-v)}\, dv\right) ds.$

Proof. The Fourier integral expression of $f(x)$ is :

$$f(x) = \int_0^\infty (A(s) \cos sx + B(s) \sin sx)\, ds, \text{ where}$$

$$A(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \cos sv\, dv \quad \text{and} \quad B(s) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v) \sin sv\, dv$$

* **Baron Baptiste Joseph Fourier** (1768–1830) French physicist and mathematician lived and taught in Paris. Fourier was appointed as Perfect of Isère by Nepoleon in 1802. He never married.

$$\therefore \qquad f(x) = \int_0^\infty \left(\left(\frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos sv \, dv \right) \cos sx + \left(\frac{1}{\pi} \int_{-\infty}^\infty f(v) \sin sv \, dv \right) \sin sx \right) ds$$

$$= \frac{1}{\pi} \int_0^\infty \left(\int_{-\infty}^\infty f(v) [\cos sv \cos sx + \sin sv \sin sx] \, dv \right) ds$$

$$= \frac{1}{\pi} \int_0^\infty \left(\int_{-\infty}^\infty f(v) \cos (sv - sx) \, dv \right) ds$$

$$\therefore \qquad f(x) = \frac{1}{\pi} \int_0^\infty \left(\int_{-\infty}^\infty f(v) \cos s(x - v) \, dv \right) ds \qquad ...(1)$$

The function $f(v) \cos s(x - v)$ is an even function of s.

$\therefore$ The function $\int_{-\infty}^\infty f(v) \cos s(x - v) \, dv$ is also an even function of s.

($\because$ The integration is not w.r.t. s)

$$\therefore \quad (1) \quad \Rightarrow \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(v) \cos s(x - v) \, dv \right) ds \qquad ...(2)$$

$$\left(\because \text{ If } \phi(x) \text{ is an even function of } x \text{ then } \int_{-\infty}^\infty \phi(x) \, dx = 2 \int_0^\infty \phi(x) \, dx \right)$$

The function $f(v) \sin s(x - v)$ is an odd function of s.

$\therefore$ The function $\int_{-\infty}^\infty f(v) \sin s(x - v) \, dv$ is also an odd function of s.

$$\therefore \qquad \int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(v) \sin s(x - v) \, dv \right) ds = 0$$

$$\Rightarrow \qquad 0 = \frac{1}{2\pi} \int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(v) \sin s(x - v) \, dv \right) ds \qquad ...(3)$$

Adding (2) and 'i' times (3), we get

$$f(x) + i \, . \, 0 = \frac{1}{2\pi} \int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(v) \left(\cos s(x - v) + i \sin s(x - v) \right) dv \right) ds$$

$$\Rightarrow \qquad f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(v) \, e^{is(x - v)} \, dv \right) ds^*.$$

$\therefore$ The result holds.

5.3. FOURIER TRANSFORM AND ITS INVERSE

Let f be a real valued function of the real variable x such that $f(x)$ and $f'(x)$ are piecewise continuous in every finite interval and the integral of $|f(x)|$ exists from $-\infty$ to ∞.

The integral $\int_{-\infty}^\infty f(x) \, e^{-isx} \, dx$ is called the **Fourier transform** of the function $f(x)$ and we write this as $F(f(x))$ or as $\bar{f}(s)$.

***Why this step.** We have used the **Euler formula :**

$$e^{ix} = \cos x + i \sin x.$$

Thus, $$\bar{\mathbf{f}}(\mathbf{s}) = \mathbf{F}(f(\mathbf{x})) = \int_{-\infty}^{\infty} \mathbf{f}(\mathbf{x})\, \mathbf{e}^{-\mathbf{isx}}\, \mathbf{dx}.$$

By Fourier integral expression of $f(x)$, we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(v)\, e^{is(x-v)}\, dv \right) ds.$$

$$\Rightarrow \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(v)\, e^{-isv}\, dv \right) e^{isx}\, ds$$

$$\Rightarrow \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx}\, ds.$$

The function $\frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx}\, ds$ *i.e.*, $f(x)$ is called the **inverse Fourier transform** of the function $\bar{f}(s)$ and we write $F^{-1}(\bar{f}(s)) = F^{-1}(F(f(x)) = f(x)$.

The constants 1 and $\frac{1}{2\pi}$ preceding the integrals $\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$ and $\frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx}\, ds$ could be replaced by any constants whose product is $\frac{1}{2\pi}$. In our discussion, we shall stick to the above choice.

Remark 1. The Fourier transform of a function will in general be a complex-valued function.

Remark 2. In finding the Fourier transform of a function, we may require the following 'general rule of integration by parts' :

$$\int \mathbf{uv\, dx} = \mathbf{uv_1 - u'v_2 + u''v_3 - u'''\, v_4 + \ldots\ldots},$$

where dashes denote the successive differentiation and suffixes denote the successive integration.

5.4. SHIFTING PROPERTY OF FOURIER TRANSFORMS

Theorem I. *If the Fourier transform of the function f(x) is* $\bar{f}(s)$ *then prove that the Fourier transform of the function f(x – a) is* $e^{-ias}\, \bar{f}(s)$.

Proof. We have

$$F(f(x)) = \bar{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx.$$

$$\therefore \quad F(f(x-a)) = \int_{-\infty}^{\infty} f(x-a)\, e^{-isx}\, dx$$

Let $t = x - a. \quad \therefore \quad dt = dx$

$$\therefore \quad F(f(x-a)) = \int_{-\infty}^{\infty} f(t)\, e^{-is(t+a)}\, dt = \int_{-\infty}^{\infty} f(t)\, (e^{-ist}\ .\ e^{-ias})\, dt$$

$$= e^{-ias} \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt = e^{-ias}\, \bar{f}(s).$$

Theorem II. *If the Fourier transform of the function f(x) is* $\bar{f}(s)$ *then prove that the Fourier transform of the function* $e^{iax} f(x)$ *is* $\bar{f}(s-a)$.

Proof. We have $\mathrm{F}(f(x)) = \bar{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx.$

$$\therefore \qquad \mathrm{F}(e^{iax} f(x)) = \int_{-\infty}^{\infty} (e^{iax} f(x))\, e^{-isx}\, dx = \int_{-\infty}^{\infty} f(x)\, e^{-i(s-a)x}\, dx$$

$$= \bar{f}(s-a).$$

5.5. MODULATION PROPERTY OF FOURIER TRANSFORMS

Theorem. *If the Fourier transform of the function f(x) is* $\bar{f}(s)$ *then prove that the Fourier transform of the function f(x) cos ax is* $\frac{1}{2}\bar{f}(s-a) + \frac{1}{2}\bar{f}(s+a)$.

Proof. We have $\mathrm{F}(f(x)) = \bar{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx.$

$$\therefore \qquad \mathrm{F}(f(x)\cos ax) = \int_{-\infty}^{\infty} (f(x)\cos ax)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{\infty} f(x) \left(\frac{e^{iax} + e^{-iax}}{2} \right) e^{-isx}\, dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} f(x)\, e^{iax}\, e^{-isx}\, dx + \frac{1}{2} \int_{-\infty}^{\infty} f(x)\, e^{-iax}\, e^{-isx}\, dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} f(x)\, e^{-i(s-a)x}\, dx + \frac{1}{2} \int_{-\infty}^{\infty} f(x)\, e^{-i(s+a)x}\, dx$$

$$= \frac{1}{2}\bar{f}(s-a) + \frac{1}{2}\bar{f}(s+a).$$

5.6. CONVOLUTION THEOREM

The convolution theorem is used to find the inverse Fourier transform of the product of two functions with known inverse Fourier transforms of the factors of the product. This theorem is also used in solving partial differential equations.

The **convolution** of the functions $f(x)$ and $g(x)$ is defined by the integral $\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$ and denoted by $f * g$.

$$\therefore \qquad (f * g)(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \qquad \text{...(1)}$$

Let $v = x - u$. $\therefore$ $dv = -du$

$$\therefore \quad (1) \Rightarrow \qquad (f * g)(x) = \int_{\infty}^{-\infty} f(x-v)\, g(v)\, (-dv)$$

$$= -\int_{\infty}^{-\infty} g(v)\, f(x-v)\, dv$$

$$= \int_{-\infty}^{\infty} g(v)\, f(x-v)\, dv = \int_{-\infty}^{\infty} g(u)\, f(x-u)\, du$$

(Replacing v by u)

$$= (g * f)(x)$$

$$\therefore \qquad f * g = g * f.$$

Convolution Theorem. Let f(x) and g(x) be functions such that their Fourier transforms exist. Prove that

$$\mathbf{F(f * g) = F(f)\, F(g).}$$

Proof. It can be proved mathematically that the Fourier transform of the convolution of f and g (*i.e.*, $f * g$) exists.

By definition, $\quad F(f * g) = \int_{-\infty}^{\infty} (f*g)(x)\, e^{-isx}\, dx$

$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \right) e^{-isx}\, dx$$

$$= \int_{-\infty}^{\infty} f(u) \left(\int_{-\infty}^{\infty} g(x-u)\, e^{-isx}\, dx \right) du$$

(By changing the order of integration)

Let $\quad v = x - u. \quad \therefore \quad dv = dx$

$$\therefore \quad \int_{-\infty}^{\infty} g(x-u)\, e^{-isx}\, dx = \int_{-\infty}^{\infty} g(v)\, e^{-is(u+v)}\, dv$$

$$\therefore \qquad F(f * g) = \int_{-\infty}^{\infty} f(u) \left(\int_{-\infty}^{\infty} g(v)\, e^{-is(u+v)}\, dv \right) du$$

$$= \int_{-\infty}^{\infty} f(u) \left(\int_{-\infty}^{\infty} g(v)\, e^{-isu}\, e^{-isv}\, dv \right) du$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{-isu} \left(\int_{-\infty}^{\infty} g(v)\, e^{-isv}\, dv \right) du$$

$$= \left(\int_{-\infty}^{\infty} f(u)\, e^{-isu}\, du \right) \left(\int_{-\infty}^{\infty} g(v)\, e^{-isv}\, dv \right)$$

$$= F(f)\, F(g)$$

$$\therefore \qquad \mathbf{F(f * g) = F(f)\, F(g).}$$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. *If* $A(s) = \dfrac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos sv\, dv$ *and* $B(s) = \dfrac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin sv\, dv$ *then*

$$f(x) = \int_0^{\infty} (A(s) \cos sx + B(s) \sin sx)\, ds.$$

Rule II. $f(x) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(v)\, e^{is(x-v)}\, dv \right) ds.$

Rule III. *The Fourier transform of the function f(x) is given by :*

$$\bar{f}(s) = F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx.$$

Rule IV. *The inverse Fourier transform of the function g(s) is given by*

$$F^{-1}(g(s)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(s)\, e^{isx}\, ds.$$

Rule V. $F^{-1}(\bar{f}(s)) = f(x)$.

Rule VI. (*i*) $F(f(x)) = \bar{f}(s) \Rightarrow F(f(x-a)) = e^{-ias}\,\bar{f}(s)$

(*ii*) $F(f(x)) = \bar{f}(s) \Rightarrow F(e^{iax} f(x)) = \bar{f}(s-a)$.

Rule VII. $F(f(x)) = \bar{f}(s) \Rightarrow F(f(x)\cos ax) = \frac{1}{2}\bar{f}(s-a) + \frac{1}{2}\bar{f}(s+a)$.

Rule VIII. (*i*) $(f * g)(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$

(*ii*) $F(f * g) = F(f)\, F(g)$.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Fourier transform of the function*

$$f(x) = \begin{cases} \lambda, & \text{if } 0 < x < a \\ 0, & \text{otherwise} \end{cases}.$$

Sol.

$$F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{0} f(x)\, e^{-isx}\, dx + \int_{0}^{a} f(x)\, e^{-isx}\, dx + \int_{a}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{0} 0\,.\, e^{-isx}\, dx + \int_{0}^{a} \lambda e^{-isx}\, dx + \int_{a}^{\lambda} 0\,.\, e^{-isx}\, dx$$

$$= 0 + \lambda \int_{0}^{a} e^{-isx}\, dx + 0 = \lambda \left(\frac{e^{-isx}}{-is} \Bigg|_{0}^{a} \right) = \mathbf{\frac{i\lambda}{s}(e^{-isa} - 1).}$$

Example 2. *Find the Fourier transform of the function*

$$f(x) = \begin{cases} e^{x}, & \text{if } -a < x < a \\ 0, & \text{otherwise} \end{cases}.$$

Sol.

$$F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{-a} f(x)\, e^{-isx}\, dx + \int_{-a}^{a} f(x)\, e^{-isx}\, dx + \int_{a}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{-a} 0\,.\, e^{-isx}\, dx + \int_{-a}^{a} e^{x}\, e^{-isx}\, dx + \int_{a}^{\infty} 0\,.\, e^{-isx}\, dx$$

$$= 0 + \int_{-a}^{a} e^{(1-is)x}\, dx + 0$$

$$= \frac{e^{(1-is)x}}{1-is} \Bigg|_{-a}^{a} = \mathbf{\frac{1}{1-is}\,[e^{(1-is)a} - e^{-(1-is)a}].}$$

Example 3. *Find the Fourier transform of the function*

$$f(x) = \begin{cases} 5x, & \text{if } 0 < x < 3 \\ 0, & \text{otherwise} \end{cases}.$$

Sol.

$$F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{0} f(x)\, e^{-isx}\, dx + \int_{0}^{3} f(x)\, e^{-isx}\, dx + \int_{3}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{0} 0\,.\, e^{-isx}\, dx + \int_{0}^{3} 5xe^{-isx}\, dx + \int_{3}^{\infty} 0\,.\, e^{-isx}\, dx$$

$$= 0 + 5 \int_{0}^{3} \underset{\text{I}}{x}\,.\,\underset{\text{II}}{e^{-isx}}\, dx + 0$$

$$= 5\left[x\, \frac{e^{-isx}}{-is}\Bigg|_{0}^{3} - \int_{0}^{3} 1\,.\, \frac{e^{-isx}}{-is}\, dx \right] = \frac{5i}{s}\,[3e^{-3is} - 0] + \frac{5}{is}\cdot\frac{e^{-isx}}{-is}\Bigg|_{0}^{3}$$

$$= \frac{15ie^{-3is}}{s} + \frac{5}{s^2}\,[e^{-3is} - 1]$$

$$= \frac{5}{s^2}\,[3is\, e^{-3is} + e^{-3is} - 1] = \mathbf{\frac{5}{s^2}\,[e^{-3is}\,(3is + 1) - 1].}$$

Example 4. *Find the Fourier transform of the function*

$$f(x) = \begin{cases} xe^{-x}, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}.$$

Sol. $\qquad F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$

$$= \int_{-\infty}^{0} f(x)\, e^{-isx} dx + \int_{0}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{0} 0\,.\, e^{-isx} dx + \int_{0}^{\infty} xe^{-x}\,.\, e^{-isx}\, dx = 0 + \int_{0}^{\infty} \underset{\text{I}}{x}\,.\,\underset{\text{II}}{e^{-(1+is)x}}\, dx$$

$$= \left[x\,.\,\frac{e^{-(1+is)x}}{-(1+is)} - 1\,.\,\frac{e^{-(1+is)x}}{(-(1+is))^2} \right]\Bigg|_{0}^{\infty}$$

$$= -\frac{1}{1+is}\,.\,\frac{x}{e^x(\cos sx + i\sin sx)}\Bigg|_{0}^{\infty} - \frac{1}{(1+is)^2}\,.\,\frac{1}{e^x(\cos sx + i\sin sx)}\Bigg|_{0}^{\infty}$$

$$= -\frac{1}{1+is}\left[\lim_{x\to\infty} \frac{x}{e^x}\,.\,\lim_{x\to\infty} \frac{1}{\cos sx + i\sin sx} - \frac{0}{e^0(\cos 0 + i\sin 0)} \right]$$

$$- \frac{1}{(1+is)^2}\left[\lim_{x\to\infty} \frac{1}{e^x(\cos sx + i\sin sx)} - \frac{1}{e^0(\cos 0 + i\sin 0)} \right]$$

$$= -\frac{1}{1+is}\,[0 - 0] - \frac{1}{(1+is)^2}\,[0 - 1]$$

$$\left(\because \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{(x)'}{(e^x)'} = \lim_{x\to\infty} \frac{1}{e^x} = 0 \right)$$

$$= \mathbf{\frac{1}{(1+is)^2}}.$$

Example 5. *Find the Fourier transform of the function*

$$f(x) = \begin{cases} -1, & \text{if} \quad -a < x < 0 \\ 1, & \text{if} \quad 0 < x < a \\ 0, & \text{otherwise} \end{cases}.$$

Sol.
$$F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{-a} f(x)\, e^{-isx}\, dx + \int_{-a}^{0} f(x)\, e^{-isx}\, dx + \int_{0}^{a} f(x)\, e^{-isx}\, dx + \int_{a}^{\infty} f(x) e^{-isx}\, dx$$

$$= \int_{-\infty}^{-a} 0\,.\, e^{-isx}\, dx + \int_{-a}^{0} -1\,.\, e^{-isx}\, dx + \int_{0}^{a} 1\,.\, e^{-isx}\, dx + \int_{a}^{\infty} 0\,.\, e^{-isx}\, dx$$

$$= 0 + (-1)\,.\, \left.\frac{e^{-isx}}{-is}\right|_{-a}^{0} + \left.\frac{e^{-isx}}{-is}\right|_{0}^{a} + 0$$

$$= -\frac{i}{s}\,(1 - e^{isa}) + \frac{i}{s}(e^{-isa} - 1)$$

$$= \frac{i}{s}\,(-1 + e^{isa} + e^{-isa} - 1) = \frac{i}{s}\,(2\cos sa - 2)$$

$$= \mathbf{\frac{2i}{s}\,(\cos sa - 1).}$$

Example 6. *Find the Fourier transform of the function*

$$f(t) = \begin{cases} 1 - \dfrac{t}{a}, & \text{if} \quad 0 < t < a \\ 1 + \dfrac{t}{a}, & \text{if} \quad -a < t < 0 \\ 0, & \text{otherwise} \end{cases}.$$

Sol. $F(f(t)) = \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$

$$= \int_{-\infty}^{-a} f(t)\, e^{-ist}\, dt + \int_{-a}^{0} f(t)\, e^{-ist}\, dt + \int_{0}^{a} f(t)\, e^{-ist}\, dt + \int_{a}^{\infty} f(t)\, e^{-ist}\, dt$$

$$= \int_{-\infty}^{-a} 0\,.\, e^{-ist}\, dt + \int_{-a}^{0} \left(1 + \frac{t}{a}\right) e^{-ist}\, dt + \int_{0}^{a} \left(1 - \frac{t}{a}\right) e^{-ist}\, dt + \int_{a}^{\infty} 0.\, e^{-ist}\, dt$$

$$= 0 + \left.\left(\left(1 + \frac{t}{a}\right).\frac{e^{-ist}}{-is} - \frac{1}{a}\,.\,\frac{e^{-ist}}{(-is)^2}\right)\right|_{-a}^{0} + \left.\left(\left(1 - \frac{t}{a}\right).\frac{e^{-ist}}{-is} - \left(-\frac{1}{a}\right).\frac{e^{-ist}}{(-is)^2}\right)\right|_{0}^{a} + 0$$

$$= -\frac{1}{is}\,(1 - 0) + \frac{1}{as^2}\,(1 - e^{ias}) - \frac{1}{is}\,(0 - 1) - \frac{1}{as^2}\,(e^{-ias} - 1)$$

$$= \frac{1}{as^2}\,(1 - e^{ias} - e^{-ias} + 1) = \frac{1}{as^2}\,(2 - 2\cos as) = \frac{1}{as^2}\,.\,4\sin^2\frac{as}{2}$$

$$= \mathbf{\frac{4}{as^2}\sin^2\frac{as}{2}}.$$