(2) let $f = \theta \longleftrightarrow \|f\|_\infty = \sup_{t\in[a,b]} \{ |f(t)|$

$\longleftrightarrow \|f\|_\infty = \sup_{t\in[a,b]} \{|\theta(t)|$

$\longleftrightarrow \|f\|_\infty = \sup_{t\in[a,b]} \{|0|\}$

$\longleftrightarrow \|f\|_\infty = 0$

(3) $\|\alpha f\|_\infty = \sup_{t\in[a,b)} \{ |\alpha f(t)| \}$

$= \sup_{t\in[a,b]} \{ |\alpha| |f(t)| \}$

$= |\alpha| \sup_{t\in[a,b)} \{ |f(t)| \}$

$= |\alpha| \, \|f\|_\infty$ $\quad \forall \alpha \in R$

$\forall f \in C[a,b]$

(4) $\|f+g\|_\infty = \sup_{t\in[a,b]} \{|(f+g)(t)|$

$= \sup_{t\in[a,b]} \{| f(t)+g(t)|\}$

$\leq \sup_{t\in[a,b]} \{|f(t)| + |g(t)|\}$

$= \sup_{t\in[a,b]} \{ |f(t)|\} + \sup_{t\in[a,b]} \{|g(t)|$

$= \|f\|_\infty + \|g\|_\infty$

$\forall f, g \in C[a,b]$

$\therefore$ $(C[a,b], \|\cdot\|_\infty)$ is normed spac

(11)

Some properties of normed space

Let $(X, \|\cdot\|)$ is a normed space over field K. Then

① $\|-x\| = \|x\|$

② $\|x-y\| = \|y-x\|$

③ $\big|\,\|x\|-\|y\|\,\big| \leq \|x-y\|$

proof: ① Let $x \in X$

$$\|-x\| = \|(-1)\,x\| \qquad \because -1 \in K$$
$$= |-1|\,\|x\|$$
$$= 1 \cdot \|x\|$$
$$= \|x\|$$

$\therefore \|-x\| = \|x\|$

② Let $x, y \in X$

$$\|x-y\| = \|(-1)(-x+y)\|$$
$$= \|(-1)(y-x)\|$$
$$= |-1|\,\|y-x\|$$
$$= 1 \cdot \|y-x\|$$
$$= \|y-x\|$$

$\therefore \|x-y\| = \|y-x\|$

(3) $x = (x-y) + y$

$\|x\| = \|(x-y)+y\|$

$\|x\| \leq \|x-y\| + \|y\|$

$\|x\| - \|y\| \leq \|x-y\|$ —— (1)

$y = (y-x) + x$

$\|y\| = \|(y-x)+x\|$

$\|y\| \leq \|y-x\| + \|x\|$

$\|y\| - \|x\| \leq \|y-x\|$ $\because \|y-x\| = \|x-y\|$ from (2)

$-(\|x\| - \|y\|) \leq \|x-y\|$ —— (2)

from (1) and (2) we get that

$\big|\|x\| - \|y\|\big| \leq \|x-y\|$.

Normed space as Metric space

The norm $\|\cdot\|$ in a ~~vector~~ normed space X over field K can be represent as metric function on K as shown bellow.

$$d(x,y) = \|x-y\| \quad \text{for all } x,y \in X$$

Metric space

Definition

Let X be any nonempty set a mapping $d: X \times X \longrightarrow R$ is said to be distance function or metric function if

1. $d(x,y) \geq 0$
2. $d(x,y) = 0$ if and only if $x = y$
3. $d(x,y) = d(y,x)$
4. $d(x,y) \leq d(x,z) + d(z,y)$

for all $x, y, z \in X$

Then (X, d) is said to be metric space.

proposition

Every normed space is a metric space.

proof: Let $(X, \|\cdot\|)$ be a normed space

we define $d(x,y) = \|x-y\|$ for any $x,y \in X$

① $d(x,y) = \|x-y\| \geq 0 \quad \forall x,y \in X$

② $d(x,y) = 0 \iff \|x-y\| = 0$

$\iff x-y=0$

$\iff x=y$

③ $d(x,y) = \|x-y\| = \|y-x\| = d(y,x)$

④ $d(x,y) = \|x-y\| = \|x-z+z-y\|$

$\leq \|x-z\| + \|z-y\|$

$= d(x,z) + d(z,y)$

$\forall x,y,z \in X$

so (X,d) is a metric space.

Lemma

A distance or (metric) function d induced by a norm on a normed space X satisfies

① $d(x+a, y+a) = d(x,y)$

② $d(\alpha x, \alpha y) = |\alpha|\, d(x,y)$

for all $a, x, y \in X$, $\alpha \in K$

proof: let $x, a, y \in X$

$$d(x+a, y+a) = \|(x+a) - (y+a)\|$$
$$= \|x + \cancel{a} - y - \cancel{a}\|$$
$$= \|x - y\|$$
$$= d(x,y)$$

let $\alpha \in K$

$$d(\alpha x, \alpha y) = \|\alpha x - \alpha y\|$$
$$= \|\alpha(x-y)\|$$
$$= |\alpha|\, \|x-y\|$$
$$= |\alpha|\, d(x,y)$$

$\forall x, y \in X$, $\forall \alpha \in K$

Sequence Space S

The space consists of the set of all (bounded or unbounded) sequences of complex numbers and the distance function d, defined by

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|}$$

$x = (x_i)_{i=1}^{\infty}$, $y = (y_i)_{i=1}^{\infty}$

$x, y \in S$ which is sequence space

(S, d) is a metric space.

proof. Let $x, y \in S$

① $d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|} \geq 0$

$\forall x, y \in S$

② $d(x,y) = 0 \Longleftrightarrow \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|} = 0$

$\Longleftrightarrow \frac{1}{2^i} \frac{|x_i - y_i|}{1 + |x_i - y_i|} = 0 \quad \forall\, 1 \leq i < \infty$

$\Longleftrightarrow |x_i - y_i| = 0$

$\Longleftrightarrow x_i - y_i = 0$

$\Longleftrightarrow x_i = y_i \quad \forall\, 1 \leq i \leq \infty$

$\Longleftrightarrow$ i.e. $x = y$

③ $$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1+|x_i - y_i|}$$
$$= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|y_i - x_i|}{1+|y_i - x_i|}$$
$$= d(y,x)$$

④ for proving ~~t~~raingular inquality consider the function f defined on $\mathbb{R}$ by $f(t) = \frac{t}{1+t}$

differentiation gives $f'(t) = \frac{1}{(1+t)^2}$

which is always positive, hence f is monotone increasing. consequently
$$|a+b| \leq |a| + |b|$$
implies
$$f(|a+b|) \leq f(|a|+|b|)$$
writing this out and applying the triangle inequality for numbers we have

$$\frac{|a+b|}{1+|a+b|} \leq \frac{|a|+|b|}{1+|a|+|b|}$$

$$= \frac{|a|}{1+|a|+|b|} + \frac{|b|}{1+|a|+|b|}$$

$$\leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$$

In this inequality we let $a = x_i - z_i$

$b = z_i - y_i$

$a+b = x_i - y_i$

where $z = (z_i)_{i=1}^{\infty}$

we have

$$\Leftrightarrow \frac{|x_i - y_i|}{1+|x_i - y_i|} \leq \frac{|x_i - z_i|}{1+|x_i - z_i|} + \frac{|z_i - y_i|}{1+|z_i - y_i|}$$

multiplying by $\frac{1}{2^i}$ and tak the summation

$$d(x,y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - y_i|}{1+|x_i - y_i|} \leq \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i - z_i|}{1+|x_i - z_i|} + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|z_i - y_i|}{1+|z_i - y_i|}$$

$\forall x, y, z \in S$ $= d(x,z) + d(z+y)$

$\therefore (S, d)$ is metric space.

Note Can every metric function or distance on a vector space be obtained from a norm? no

The answer is no, The following example is show that.

Example The ~~space~~ ~~vector~~ space (S,d) is metric but can not be obtained from a norm.

As we see from the lemma the metric function d induced by norm satisfy two condition

① $$d(x+a,\ y+a) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|(x_i+a_i)-(y_i+a_i)|}{1+|(x_i+a_i)-(y_i+a_i)|}$$
$$= \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i-y_i|}{1+|x_i-y_i|}$$
$$= d(x,y)$$

② $$d(\alpha x, \alpha y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|\alpha x_i-\alpha y_i|}{1+|\alpha x_i-\alpha y_i|}$$
$$= |\alpha| \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|x_i-y_i|}{1+|\alpha|\,|x_i-y_i|}$$
$$\neq |\alpha|\, d(x,y)$$

Sequence in Vector space.

Let X be a vector space over a field K and N be the set of natural number

The mapping from N into X is called sequence in X

$f: N \longrightarrow X$ such that

$f(n) = x_n \quad n \in N$

So for each $n \in N$, there exists one element $x_n \in X$, we denoted to sequence by $(x_n)_{n=1}^{\infty}$ or $(x_n)_{n \in N}$ or simply by (x_n)

Example Let R^2 be a vector space over R

$x_n = (n, \frac{1}{n})$

So $(x_n)_{n=1}^{\infty}$ is a sequence in R^2

$y_n = (2, \frac{1}{n^2})$

$(y_n)_{n=1}^{\infty}$ is a sequence in R^2

Convergent sequence

Let $(x_n)_{n=1}^{\infty}$ be a sequence in a normed space $(X, \|\cdot\|)$. The sequence $(x_n)_{n=1}^{\infty}$ is said to be convergent if there exists $x_0 \in X$, such that for all $\epsilon > 0$ there exists $k \in N$ such that

$$\|x_n - x_0\| < \epsilon \qquad \forall n > k$$

Note (1)

The convergent in another word, all the sequence term has distance less than ϵ from the convergent point except the first k-terms.

Example Let R^2 be ~~vector~~ normed space over R with $\|x\| = \|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}$

$e\ x_n = (\frac{1}{n}, \frac{1}{n})$, $(x_n)_{n=1}^{\infty} \to \theta = (0, 0)$

$$\|x_n - \theta\| = \|(\tfrac{1}{n}, \tfrac{1}{n}) - (0,0)\| = \|(\tfrac{1}{n}, \tfrac{1}{n})\|$$

$$= \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\sqrt{2}}{n} < \epsilon$$

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$\therefore (x_n)_{n=1}^{\infty} \to \theta$

is convergent sequence.

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Let a, b areal number and $a > 0$ the there exist $n \in N$ such that

$$na > b$$

or $$a > \frac{b}{n}$$

Corollarly

for any real number $\epsilon > 0$, there exist $n \in N$ such that

$$\frac{1}{n} < \epsilon$$

Note 2

If the sequence $(X_n)_{n=1}^{\infty}$ convergent to X_0 we always denoted it by

$$X_n \to X_0$$

Some times we write it as $\lim_{n \to \infty} X_n = X_0$

If $(X_n)_{n=1}^{\infty}$ is not convergent then it is said to be divergent.

Theorem

Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence in a normed space $(\mathcal{X}, \|\cdot\|)$. Then the convergent point is unique.

proof: Let $(x_n)_{n=1}^{\infty}$ sequence in $\mathcal{X}$ which is convergent to x_0 and y_0 that is mean

$$x_n \longrightarrow x_0$$
$$x_n \longrightarrow y_0$$

and $x_0 \neq y_0$ we have different convergent point

$\because x_n \to x_0$, $\exists\, k_1 \in N$ such that

$$\|x_n - x_0\| < \frac{\epsilon}{2} \qquad \forall n > k_1$$

$\because x_n \longrightarrow y_0$, $\exists\, k_2 \in N$ such that

$$\|x_n - y_0\| < \frac{\epsilon}{2} \qquad \forall n > k_2$$

Let $K = \max\{k_1, k_2\}$. Then

$$\begin{aligned}
\epsilon = \|x_0 - y_0\| &= \|x_0 - x_n + x_n - y_0\| \\
&= \|-(x_n - x_0) + (x_n - y_0)\| \\
&\leq \|-(x_n - x_0)\| + \|x_n - y_0\| \\
&= \|x_n - x_0\| + \|x_n - y_0\| \\
&= \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \forall n > k
\end{aligned}$$

we get $\epsilon < \epsilon$ which is contradiction so must $x_0 = y_0$

proposition

Let $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ be two convergent sequence in normed space $(X, \|\cdot\|)$ such that

$$x_n \longrightarrow x_0$$
$$y_n \longrightarrow y_0 \qquad x_0, y_0 \in X$$

Then

① $(x_n + y_n) \longrightarrow x_0 + y_0$

② $\alpha x_n \longrightarrow \alpha x_0 \qquad \alpha \in K$

③ $\|x_n\| \longrightarrow \|x_0\|$

④ $\|x_n - y_n\| \longrightarrow \|x_0 - y_0\|$

proof. ① $\because x_n \longrightarrow x_0$, for $\frac{\epsilon}{2} > 0$, there exist $k_1 \in N$ such that

$$\|x_n - x_0\| < \frac{\epsilon}{2} \qquad \forall n > k_1$$

$\because y_n \longrightarrow y_0$, for $\frac{\epsilon}{2} > 0$, there exist $k_2 \in N$ such that

$$\|y_n - y_0\| < \frac{\epsilon}{2} \qquad \forall n > k_2.$$

Let $k = \max\{k_1, k_2\}$

$$\begin{aligned}\|(x_n + y_n) - (x_0 - y_0)\| &= \|(x_n - x_0) + (y_n - y_0)\| \\ &\leq \|x_n - x_0\| + \|y_n - y_0\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \forall n > k\end{aligned}$$

$\therefore (x_n + y_n) \longrightarrow x_0 + y_0$