2.3 Self-adjoint, unitary and normal operators

Definition A bounded linear operator $T: H \longrightarrow H$ on a Hilbert space H is said to be

self-adjoint if $T^* = T$

unitary if T is bijective and $T^* = T^{-1}$

normal if $TT^* = T^*T$

The Hilbert adjoint operator T^* defined by

$$\langle T(x) / y \rangle = \langle x / T^*(y) \rangle$$

~~$\forall x \in$~~ $\forall x, y \in H$.

Note If T is self adjoint that means

$$\langle T(x) / y \rangle = \langle x / T(y) \rangle$$

If T is unitary then

$$\langle T(x) / y \rangle = \langle x / T^{-1}(y) \rangle$$

and T is bijective

or $\langle T(x) / T(y) \rangle = \langle x / y \rangle$

Lemma Let $T: H \to H$ be an bounded linear operator

① if T is self adjoint then T is normal

② if T is unitary then T is normal

proof ① Let T is self adjoint then

$$T = T^*$$

$$\therefore T T^* = (T^*) T \qquad \because T = T^*$$

$\therefore$ T is ~~unitary~~ normal.

proof ② Let T is unitary then $T^* = T^{-1}$

$$T T^* = T(T^{-1}) = I$$

$$T^* T = T^{-1}(T) = I$$

so $T T^* = T^* T$

$\therefore$ T is normal.

Example Let $I: H \to H$ be the identity operator on Hilbert space H, then I is self adjoint operator. It is clear that I linear bounded $x, y \in H$

$$\langle I(x)/y \rangle = \langle x / I^*(y) \rangle$$

$$\langle x/y \rangle = \langle x / I^*(y) \rangle$$

$$\langle x/y \rangle - \langle x / I^*(y) \rangle = 0$$

$\langle x / y - I^{*}(y) \rangle = 0$

$\langle x / I(y) - I^{*}(y) \rangle = 0$

$\langle x / (I - I^{*})(y) \rangle = 0 \quad \forall x, y \in H.$

by zero lemma we have

$I - I^{*} = 0$

$I = I^{*}$

$\therefore I$ is self adjoint operator.

hence $I = I^{-1}$

So $I^{*} = I = I^{-1}$

$\therefore I^{*} = I^{-1}$

and I is bijective

So I is unitary

Example Let $O : H \longrightarrow H$ the zero operator on Hilbert space H, then O is self adjoint operator.

It is clear that O is bounded linear $\forall x, y \in H$

$\langle O(x) / y \rangle = \langle x / O^{*}(y) \rangle$

$\langle 0 / y \rangle = \langle x / O^{*}(y) \rangle$

$\langle 0 . x / y \rangle = \langle x / O^{*}(y) \rangle$

$\langle x / \bar{0} y \rangle = \langle x / O^{*}(y) \rangle$

$\langle x / 0 . y \rangle = \langle x / O^{*}(y) \rangle$

$\langle x / O(y) \rangle = \langle x / O^{*}(y) \rangle$

$\langle x / O(y) \rangle - \langle x / O^{*}(y) \rangle = 0$

$\langle x / O(y) - O^{*}(y) \rangle = 0$

$\langle x / (O - O^{*})(y) \rangle = 0$

by using zero lemma, then

$$0 - 0^{*} = 0 \quad \text{so}$$

$$0^{*} = 0$$

$\therefore$ 0 is self adjoint operator.

The zero operator 0 is not one-to-one or surjective so it is not unitary

~~th~~ this is example of self adjoint operator which is not unitary.

Example we will give an example of normal operator but not self adjoint or unitary

Let $I : H \longrightarrow H$ be ~~identity~~ Identity operator ~~Then~~ ~~w~~ here H is a Hilbert space

we define an operator $T : H \longrightarrow H$ such that

$$T(x) = 2ix$$

is T self adjoint, unitary, normal operator?

At first we to show that T is bounded linear operator

Let $x, y \in H$, $\alpha, \beta \in K$, so

$$T(\alpha x + \beta y) = 2i(\alpha x + \beta y) = 2i\alpha x + 2i\beta y$$

$$= \alpha(2ix) + \beta(2iy)$$

$$= \alpha T(x) + \beta T(y)$$

$\therefore$ T is linear operator

we to show that T is bounded

Let $x \in H$

$\|T(x)\| = \| 2ix \| = |$

$= |2i| \|x\|$ $\quad \because |2i| = \sqrt{0^2 + 2^2}$

$= 2\|x\| \quad \forall \quad = 2$

$\therefore \|T(x)\| = 2\|x\| \quad \forall x \in X$

So T is bounded,

$\|T\| = \sup_{0 \neq x \in H} \left\{ \frac{\|T(x)\|}{\|x\|} \right\} = \sup_{0 \neq x \in H} \left\{ \frac{2\cancel{\|x\|}}{\cancel{\|x\|}} \right\}$

$\|T\| = 2$

now we to find the adjoint operator

$\langle T(x) / y \rangle = \langle x / T^*(y) \rangle$

$\langle 2ix / y \rangle = \langle x / T^*(y) \rangle$

$2i \langle x / y \rangle = \langle x / T^*(y) \rangle$

$\langle x / \overline{2i} y \rangle = \langle x / T^*(y) \rangle$

$\langle x / -2iy \rangle - \langle x / T^*(y) \rangle = 0$

$\langle x / -2i I(y) \rangle - \langle x / T^*(y) \rangle = 0$

$\langle x / -2i I(y) - T^*(y) \rangle = 0$

$\langle x / (-2iI - T^*)(y) \rangle = 0$ using zero lemma

$\therefore$ ~~$T^* = -2iI$~~ $\quad T^* = -2iI \quad \forall y \in H$

$T^*(y) = -2iy$

$T^*(y) = -2iy$

we see that

$T(x) = 2ix \neq -2ix = T^*(x)$

$\therefore T(x) \neq T^*(x)$

T is not self adjoint operator

$\therefore T(x) = 2ix$

let $T(x) = T(y)$

$\dfrac{2ix}{2i} = \dfrac{2iy}{2i}$

$x = y$

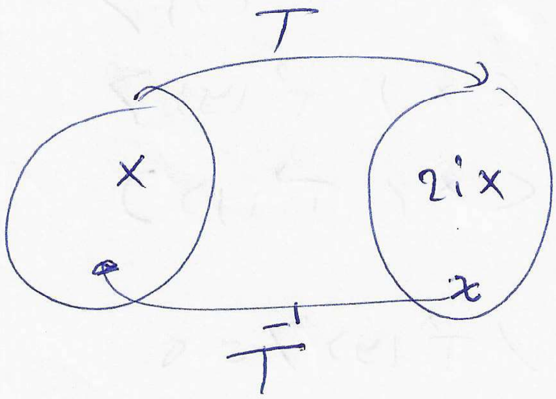
T is one-to-one

~~$T(H) = H$~~

$\forall\, x \in H \quad T(x) = 2ix$

$\therefore T$ is onto

so T is bijective



$\dfrac{i\,T(x)}{i\,2i} = \dfrac{2ix}{2i}$

$\dfrac{i}{2i^2} T(x) = x$

$-\dfrac{1}{2} T(x) = x$

$T(-\frac{1}{2} i x) = x$

so $T^{-1}(x) = -\frac{1}{2} i x$

$T^*(x) = -2ix \neq -\frac{1}{2} i x = T^{-1}(x)$

$\therefore T^*(x) \neq T^{-1}(x)$

$\therefore T$ is not unitary

now take To show that T is ~~unit~~ normal

$$TT^*(x) = T(T^*(x))$$
$$= T(-2ix)$$
$$= -2i\,T(x)$$
$$= -2i\;2ix$$
$$= -4i^2x$$
$$= 4x$$

$\therefore TT^*(x) = 4x \quad \forall x \in H$

now $T^*T(x) = T^*(T(x))$
$$= T^*(2ix)$$
$$= 2i\,T^*(x)$$
$$= 2i(-2ix)$$
$$= -4i^2x$$
$$= 4x$$

$\therefore T^*T(x) = 4x$

So $TT^*(x) = T^*T(x) \quad \forall x \in H$

$\therefore TT^* = T^*T$

So T is normal operator.

Example Let $T : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ ($\mathbb{C}^2$ is over $\mathbb{C}$) be an operator defined by $T(x) = T(x_1, x_2) = (x_2, x_1)$

find T^* if exist and which any type of self-adjoint, unitary, normal are belong?

Solution At first must show that T is bounded linear operator.

Let $\alpha, \beta \in \mathbb{C}$, $x, y \in \mathbb{C}^2$ such that $x = (x_1, x_2)$, $y = (y_1, y_2)$, $x_1, x_2, y_1, y_2 \in \mathbb{C}$

$$
\begin{aligned}
T(\alpha x + \beta y) &= T(\alpha(x_1, x_2) + \beta(y_1, y_2)) \\
&= T((\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)) \\
&= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) \\
&= (\alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1) \\
&= (\alpha x_2, \alpha x_1) + (\beta y_2, \beta y_1) \\
&= \alpha(x_2, x_1) + \beta(y_2, y_1) \\
&= \alpha T(x_1, x_2) + \beta T(y_1, y_2) \\
&= \alpha T(x) + \beta T(y)
\end{aligned}
$$

$\therefore$ T is linear operator

we to show that T is bounded

Let $x = (x_1, x_2) \in \mathbb{C}^2$ where $x_1, x_2 \in \mathbb{C}$

$$
\begin{aligned}
\|T(x)\|^2 &= \langle T(x) / T(x) \rangle \\
&= \langle T(x_1, x_2) / T(x_1, x_2) \rangle \\
&= \langle (x_2, x_1) / (x_2, x_1) \rangle \\
&= x_2 \overline{x_2} + x_1 \overline{x_1} \\
&= x_1 \overline{x_1} + x_2 \overline{x_2} = \langle (x_1, x_2) / (x_1, x_2) \rangle \\
&= \|x\|^2
\end{aligned}
$$

8