Example of Stokes's Theorem

Example 3: Use Stokes's theorem to find the work performed (الشغل المنجز) by the vector field $F(x,y,z) = x^2 i + 4xy^3 j + y^2 x\ k$ on a particle that traverses (تقطع) the rectangular C in the plane $z = y$ and the lies $x = 1$ & $y = 3$.

Solution:

$$\iint_S Curl\ F.n.ds$$

$$Curl\ F = \nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & 4xy^3 & y^2 x \end{vmatrix} = (2xy - 0)i - (y^2 - 0)j + (4y^3 - 0)k$$

$$\therefore \quad Curl\ F = 2xy\ i - y^2 j + \ 4y^3 k$$

$$n = \frac{\nabla F}{\|\nabla F\|} = \frac{0 + (-1)j + 1k}{\sqrt{0+1+1}} = \frac{-j+k}{\sqrt{2}}$$

$$ds = \sqrt{{f_x}^2 + {f_y}^2 + 1}\ dA \quad \Rightarrow \quad ds = \sqrt{0^2 + 1^2 + 1}\ dA \quad \Rightarrow ds = \sqrt{2}\, dA$$

$$= \iint_S (2xy\ i - y^2 j + \ 4y^3 k).\frac{-j+k}{\sqrt{2}}.\sqrt{2}\, dA \quad = \iint_S (y^2 + \ 4y^3)\, dA =$$

$$= \int_0^1 \int_0^3 (y^2 + \ 4y^3)\, dy\, dx \ = \int_0^1 \ \frac{y^3}{3} + y^4 \Big]_0^3 \ dx$$

Example 4: Use Stokes's theorem to find the work performed (الشغل المنجز) by the vector field $F(x, y, z) = -yi + x^2 j + z^2\, k$ on a particle that traverses (يقطع) along C, where C is occur (يحدث) by the intersection between the cylinder $x^2 + y^2 = 4$ and the plane $x + z = 3$.

Solution:

$$\iint_S Curl\ F.n.ds$$

$$Curl\ F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x^2 & z^2 \end{vmatrix} = (0-0)i - (0-0)j + (2x+1)k =$$

$$\therefore \quad Curl\ F = (2x+1)k$$

$$n = \frac{\nabla F}{\|\nabla F\|} = \frac{1i + (0)j + 1k}{\sqrt{1+0+1}} = \frac{i+k}{\sqrt{2}} \quad , ds = \sqrt{2}\, dA$$

$$= \iint_S (2x+1)k.\frac{i+k}{\sqrt{2}}.\sqrt{2}\, dA = \iint_S 2x - 1\, dA = \int_0^2 \int_0^{\sqrt{4-x^2}} 2x + 1\, dy\, dx$$

$$= \int_0^1 \left. \frac{y^3}{3} + y^4 \right]_0^3 dx$$

Example 5: Verify Stokes's theorem for field $F(x,y) = zi + xj + yk$ on the hemisphere $z = 1 - x^2 - y^2$

Solution:

$$\oint_C F.dR = \iint_S Curl\ F.n.ds$$

$$z + x^2 + y^2 - 1 = 0$$

$$\iint_S Curl\ F.n.ds$$

$$Curl\ F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = (1-0)i - (0-1)j + (1-0)k =$$

$$\therefore\ Curl\ F = i + j + k$$

$$n = \frac{\nabla F}{\|\nabla F\|} = \frac{2xi+2yj+1k}{\sqrt{4x^2+4y^2+1}} \quad , ds = \frac{\sqrt{4x^2+4y^2+1}}{1}\ dA$$

$$\iint_S (i+j+k)\,.\frac{2xi+2yj+k}{\sqrt{4x^2+4y^2+1}}.\sqrt{4x^2+4y^2+1}\ dA$$

$$= \iint_S (2x+2y+1)\ dA\ = \int_0^{2\pi}\int_0^1 (2r cos\theta + 2\ r sin\theta + 1)\ rdr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 (2r^2 cos\theta + 2\ r^2 sin\theta + r)dr\ d\theta = \int_0^{2\pi} \frac{2r^3}{3} cos\theta + \frac{2r^3}{3} sin\theta + \frac{r^2}{2}\Big]_0^1 d\theta$$

$$\int_0^{2\pi} \frac{2}{3}cos\theta + \frac{2}{3}sin\theta + \frac{1}{2}\Big]\ d\theta = \frac{2}{3}sin\theta - \frac{2}{3}\ cos\theta + \frac{1}{2}\theta\Big]_0^{2\pi} = 0 - \frac{2}{3} + \pi - [0 - 1 + 0]$$

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$$\oint_C F.dR =$$

$$\oint_C (yi - xj).(dxi + dyj + dzk) = \oint_C (ydx - xdy) = \iint_R (-1-1)dA$$

$$= -2\iint_R dA = -2\int_0^{2\pi}\int_0^a r\,dr\,d\theta = -2\int_0^{2\pi} \frac{r^2}{2}\Big]_0^a d\theta = -2a^2\pi.$$