Proposition (4.4): Let (X, d) be a metric space then every compact set is closed.

Proof: Let $A \subset X$ be a compact subset

Suppose $x, y \in X$ s.t. $x \notin A$, $y \in A$ and

$d(x, y) = r$ s.t. $B_{\frac{r}{2}}(x) \cap B_{\frac{r}{2}}(y) = \emptyset$, $\forall\, y \in A$

Since A is compact

$\Rightarrow$ $\exists$ finitely many points in A

i.e. $y_1, y_2, \dots, y_n$ in A

s.t. $A \subset B_{\frac{r_1}{2}}(y_1) \cup B_{\frac{r_2}{2}}(y_2) \cup \dots \cup B_{\frac{r_n}{2}}(y_n) = G_1$

Let $G_2 = B_{\frac{r_1}{2}}(x) \cap B_{\frac{r_2}{2}}(x) \cap \dots \cap B_{\frac{r_n}{2}}(x)$

We have $G_1 \cap G_2 = \emptyset$

$\Rightarrow$ $G_2 \subset A^c$

$\Rightarrow$ x is an interior point of A^c

$\Rightarrow$ A^c is an open set

$\Rightarrow$ A is closed set

Corollary (4.1): If F is a closed set and K is a compact set, then $F \cap K$ is a compact set.

Proof: Since K is a compact set

$\Rightarrow$ K is a closed set

Since F and K are closed sets

$\Rightarrow$ $F \cap K$ is a closed set and $F \cap K \subset K$

$\Rightarrow$ $F \cap K$ is a compact set

Proposition (4.5): Let (X, d) be a metric space then every compact set is bounded.

Proof: Let E be a compact set in X and $x_0 \in X$

Put $B_n = \{x \in X : d(x, x_0) < n\}$, $\forall\, n \in N$, B_n is an open set.

Let $x \in E$, then $\exists\, n \in N$ s.t. $d(x, x_0) < n$

$\Rightarrow\ x \in B_n$

$\Rightarrow\ E \subseteq \bigcup_n B_n$

$\Rightarrow\ \{B_n : n \in N\}$ is an open cover for E

But E is a compact set, then $\exists\ k \in N$ s.t. $E \subseteq \bigcup_{n=1}^{k} B_n$

$B_1 \subseteq B_2 \subseteq B_3 \subseteq \cdots$

Then $\bigcup_{n=1}^{k} B_n = B_k$

$\Rightarrow\ E \subseteq B_k$

$\Rightarrow\ E$ is bounded.

Theorem (4.8): (Heine – Borel Theorem) نظرية هاين - بوريل

A subset of R is compact iff it is closed and bounded.

Proof: Let $E \subset R$ be a compact set

By Propositions (4.4) and (4.5),

$\Rightarrow\ E$ is closed and bounded set

Now, let E be a closed and bounded set in R

Since E is bounded

$\Rightarrow\ \exists\ M \in R$, s.t. $|x| \le M$, $\forall\ x \in E$

$\Rightarrow\ E \subset [-M, M]$

Since $[-M, M]$ is closed interval in R

$\Rightarrow\ [-M, M]$ is compact

Since E is closed

$\Rightarrow\ E$ is compact

Connectedness in Metric Spaces الـتـرابـط فـي الـفـضـاءات الـمـتـريـة

Definition (4.17): (Separation) **الانفصال**

Let (X, d) be a metric space. We say that the set E is **separable** in (X, d), if there is two open sets A, B such that

(i) $A, B \neq \emptyset$

(ii) $A \cap B = \emptyset$

(iii) $E \subseteq A \cup B$

Example (4.20): The set $E = \{0,2\}$ is separable set in (R, d)

Since $\exists$ $A = (-1,1)$ and $B = (1,3)$ satisfy

(1) $A, B \neq \emptyset$

(2) $A \cap B = (-1,1) \cap (1,3) = \emptyset$

(3) $E = \{0,2\} \subset (-1,1) \cup (1,3) = A \cup B$

Definition (4.18): (Connectedness) **الترابط**

Let (X, d) be a metric space. We say that a subset E is connected if there does not exist a separation for E in (X, d).

Example (4.21): The set $E = [0,2]$ is connected set in (R, d), since there does not exist a separation for [0,2] in (R, d).

Proposition (4.6): A metric space (X, d) is connected if and only if the only clopen subsets of X are the empty set.

Chapter Five
الاستمرارية
Continuity

Definition (5.1): (Continuous Function) الدالة المستمرة

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \to Y$ is **continuous** at $x_0 \in X$ if

$\forall\ \varepsilon > 0, \exists\ \delta > 0, \delta = \delta(\varepsilon, x_0)$ s.t. if $d_X(x, x_0) < \delta\ \Rightarrow d_Y\big(f(x), f(x_0)\big) < \varepsilon$.

i.e. In terms of open balls, $f\big(B_\delta(x_0)\big) \subset B_\varepsilon\big(f(x_0)\big)$

Example (5.1): Let (R, d) be usual metric space then every constant function is continuous.

Proof:

We have $d(x, y) = |x - y|,\ \forall\ x, y \in R$

Let $f: (R, d) \to (R, d)$ defined by

$f(x) = c\ , \forall\ \ x \in R\ , c$ is constant

Let $\varepsilon > 0\ , \exists\ \delta > 0$, s.t. $d(x, x_0) = |x - x_0| < \delta$

$\Rightarrow\ d\big(f(x), f(x_0)\big) = |f(x) - f(x_0)|$

$= |c - c| = 0 < \varepsilon$

$\therefore\ d\big(f(x), f(x_0)\big) < \varepsilon$

$\Rightarrow\ f$ is continuous function

Example (5.2): Prove that every identity function is continuous.

Proof:

Let (R, d) be uusual metric space

Let $f: (R, d) \to (R, d)$ defined by

$f(x) = x\ , \forall\ \ x \in R$