(3) Show that if f is Riemann integrable on $[a, b]$, then f^2 also is Riemann integrable on $[a, b]$.

Chapter Three
نـظـريـة الـقـيـاس
Measure Theory

Definition (3.1): (Open Interval) الـفـتـرة الـمـفـتـوحـة

Let $a, b \in R$ and $a < b$ we define the **open interval** as

$$(a, b) = \{x \in R : a < x < b\}$$

We will denoted for an open interval by I, and consider $\emptyset$ is an open interval.

Definition (3.2): (Length of Interval) طـول الـفـتـرة

Let I be an open interval (bounded), the **length** of interval I defined as

$$\Delta(I) = \begin{cases} b - a\,, & I = (a, b) \\ 0\,, & I = \emptyset \end{cases}$$

Where $\Delta(I)$ is a measure of the length of interval I.

Note (3.1):

(1) $\Delta(I) \geq 0$.

(2) $\Delta(I) = 0 \quad \Leftrightarrow \quad I = \emptyset$.

(3) If $I_1 \subset I_2$ then $\Delta(I_1) < \Delta(I_2)$.

(4) In some references, the length of I is denoted by the $|I|$.

Example (3.1): Let $I_1 = (0,2)$, $I_2 = (-3,3)$, $I_3 = (4,9)$ and $I_4 = \emptyset$, find the lengths of these open intervals.

Solution:

$\Delta(I_1) = 2 - 0 = 2$,

$\Delta(I_2) = 3 - (-3) = 6$,

$\Delta(I_3) = 9 - 4 = 5$,

$\Delta(I_4) = 0$.

Note the

$I_1 = (0,2) \subset I_2 = (-3,3)$

and

$\Delta(I_1) = 2 < \Delta(I_2) = 6$

معدود

Theorem (3.1): Every open set G of R can be written uniquely as a countable

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union of disjoint open intervals.

Definition (3.3): If G is an open subset of $[a, b]$, $G = \bigcup_{n} I_n$, $n \in N$ then the **length** of G is defined as

$$\Delta(G) = \Delta(\bigcup_{n} I_n) = \sum_{n=1} \Delta(I_n)$$

Where $\Delta(I_n)$ denotes the length of the interval I_n, $n \in N$.

Note (3.2):

(1) $\Delta(G) \geq 0$.

(2) $\Delta(G) = 0 \quad \Leftrightarrow \quad G = \emptyset$.

(3) If $G_1 \subset G_2$ then $\Delta(G_1) < \Delta(G_2)$.

(4) $\Delta(G_1 \cup G_2) = \Delta(G_1) + \Delta(G_2) - \Delta(G_1 \cap G_2)$.

Example (3.2): Let $G_1 = \{(0,1) \cup (1,2) \cup (2,3) \cup (3,4)\}$, $G_2 = \{(-1,0) \cup (2,4)\}$, find $\Delta(G_1)$, $\Delta(G_2)$, $\Delta(G_1 \cup G_2)$ and $\Delta(G_1 \cap G_2)$.

Solution:

$$\Delta(G) = \Delta(\bigcup_{n} I_n) = \sum_{n=1}^{\infty} \Delta(I_n)$$

$$\Delta(G_1) = \sum_{n=1}^{4} \Delta(I_n) = \Delta(I_1) + \Delta(I_2) + \Delta(I_3) + \Delta(I_4)$$

$$= (1-0) + (2-1) + (3-2) + (4-3) = 4$$

$$\Delta(G_2) = \sum_{n=1}^{2} \Delta(I_n) = \Delta(I_1) + \Delta(I_2)$$

$$= (0-(-1)) + (4-2) = 3$$

Now,

$G_1 \cup G_2 = \{(-1,0) \cup (0,1) \cup (1,2) \cup (2,4)\}$,

$\Delta(G_1 \cup G_2) = (0-(-1)) + (1-0) + (2-1) + (4-2) = 5$

and

$G_1 \cap G_2 = \{(2,3) \cup (3,4)\}$

$\Delta(G_1 \cap G_2) = (3-2) + (4-3) = 2$

Note that

$$\Delta(G_1 \cup G_2) = \Delta(G_1) + \Delta(G_2) - \Delta(G_1 \cap G_2)$$

$$= 4 + 3 - 2 = 5$$

Theorem (3.2): If $G_1, G_2, G_3, \ldots$ are open subsets of $[a, b]$, then

$$\Delta\left(\bigcup_{n=1}^{\infty} G_n\right) \leq \sum_{n=1}^{\infty} \Delta(G_n) .$$

Example (3.3): Let $G_1 = \{(-2,1) \cup (1,3) \cup (4,6)\}$, $G_2 = \{(0,2) \cup (3,5) \cup (5,7)\}$, $G_3 = \{(1,3) \cup (3,4) \cup (5,6)\}$, find $\Delta(\cup_{n=1}^{3} G_n)$ and $\sum_{n=1}^{3} \Delta(G_n)$.

Solution: