Double Integrals in Polar Coordinates

التكاملات الثنائية بالإحداثيات القطبية

In Cartesian Coordinates (بالإحداثيات الديكارتي): $dA = dx\,dy = dy\,dx$

In Polar Coordinates (بالإحداثيات القطبية): $dA = r\,dr\,d\theta = r\,d\theta\,dr$

Prefer to use integrals in polar coordinates when we have polar region, also, if we have $(x^2 + y^2)$, then used polar coordinates.

$x = r\cos\theta\,, \quad y = r\sin\theta$

$x^2 + y^2 = r^2$

$(r\cos\theta)^2 + (r\sin\theta)^2 \Rightarrow r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta) = r^2$

Example 1: Evaluate the double integral by using the polar Coordinates

$\iint_R (1 + x^2 + y^2)^{\frac{3}{2}}\,dA$, where $0 \le r \le 1$, $0 \le \theta \le 2\pi$

Solution:

$$\int_0^{2\pi}\int_0^1 (1 + r^2)^{\frac{3}{2}}\,r\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi} \left.\frac{(1+r^2)^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1 d\theta = \frac{1}{5}\int_0^{2\pi}[(2)^{\frac{5}{2}} - (1)^{\frac{5}{2}}]d\theta$$

$$\left.\frac{1}{5}(2)^{\frac{5}{2}}\theta\right]_0^{2\pi} - \left.\frac{1}{5}\theta\right]_0^{2\pi} = \frac{2}{5}(2)^{\frac{5}{2}}\pi - \frac{2}{5}\pi$$

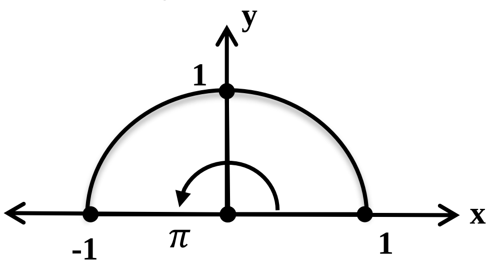
Example 2: $\int_{-1}^{1}\int_0^{\sqrt{1-x^2}} x(x^2 + y^2)^2\,dy\,dx$

Solution: $-1 \le x \le 1$ and

$y = 0,\ \ y = \sqrt{1 - x^2} \quad \Rightarrow \quad y^2 = 1 - x^2 \quad \Rightarrow \quad x^2 + y^2 = 1$

$\int_0^{\pi}\int_0^1 r\cos\theta\,(r^2)^2\,r\,dr\,d\theta =$

$$\int_0^{\pi}\int_0^1 r^6\cos\theta\,dr\,d\theta = \int_0^{\pi}\cos\theta \left.\frac{r^7}{7}\right]_0^1 d\theta = \frac{1}{7}\int_0^{\pi}\cos\theta\,d\theta = \frac{1}{7}\sin\theta\Big]_0^{\pi} = 0$$

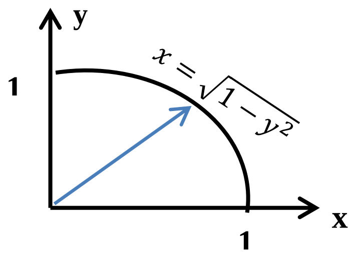


Example 3: $\int_0^1 \int_0^{\sqrt{1-y^2}} (x^2+y^2)^{\frac{1}{3}}\ dx\, dy$

Solution: $0 \le y \le 1$ and $x = 0,\ x = \sqrt{1-y^2} \Rightarrow x^2 = 1-y^2 \Rightarrow x^2+y^2 = 1$

$$\int_0^{\frac{\pi}{2}} \int_0^1 (r^2)^{\frac{1}{3}}\ r\, dr\, d\theta =$$

$$\int_0^{\frac{\pi}{2}} \int_0^1 r^{\frac{2}{3}}\, r\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \int_0^1 r^{\frac{5}{3}}\, dr\, d\theta = \int_0^{\frac{\pi}{2}} \left.\frac{r^{\frac{8}{3}}}{\frac{8}{3}}\right]_0^1 d\theta$$



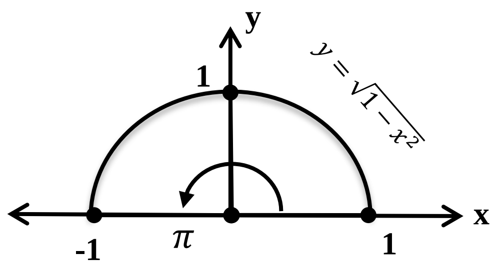
$$\frac{3}{8}\int_0^{\frac{\pi}{2}} d\theta = \frac{3}{8}\ \theta\Big]_0^{\frac{\pi}{2}} = \frac{3}{16}\pi$$

Example 4: $\int_{-1}^1 \int_0^{\sqrt{1-x^2}}\ dy\, dx$

Solution: $-1 \le x \le 1$ and $y = 0,\ y = \sqrt{1-x^2} \Rightarrow y^2 = 1-x^2 \Rightarrow x^2+y^2 = 1$

$$\int_0^{\pi} \int_0^1 r\, dr\, d\theta = \int_0^{\pi} \left.\frac{r^2}{2}\right]_0^1 d\theta$$

$$= \frac{1}{2}\int_0^{\pi} d\theta = \frac{1}{2}\ \theta\Big]_0^{\pi} = \frac{1}{2}\pi$$

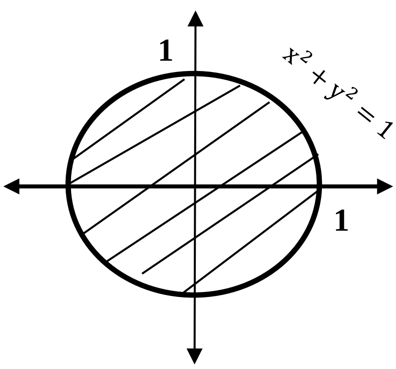


Example 5: $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (x^2+y^2)\, dx\, dy$

Solution: $-1 \le y \le 1\ ,\ x = \pm\sqrt{1-y^2} \Rightarrow x^2+y^2 = 1$

$$\int_0^{2\pi} \int_0^1 r^2\ r dr\, d\theta = \int_0^{2\pi} \int_0^1 r^3\ dr\, d\theta =$$

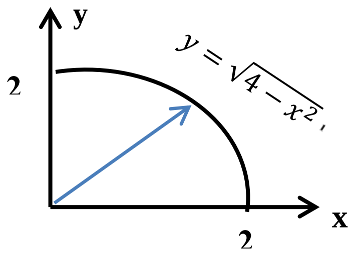
$$\int_0^{2\pi} \left.\frac{r^4}{4}\right]_0^1 d\theta = \frac{1}{4}\int_0^{2\pi} d\theta = \frac{1}{4}\ \theta\Big]_0^{2\pi} = \frac{1}{2}\pi$$



Example 6: $\int_0^2 \int_0^{\sqrt{4-x^2}} \; dy\,dx$

Solution: $0 \le x \le 2 \quad ,y = 0, \;\; y = \sqrt{4-x^2} \;\; \Rightarrow \quad \mathrm{x}^2 + \mathrm{y}^2 = 4$

$$\int_0^{\frac{\pi}{2}} \int_0^2 \; r\,\mathrm{dr}\,\mathrm{d\theta} = \int_0^{\frac{\pi}{2}} \frac{r^2}{2}\Big]_0^2 \mathrm{d\theta}$$



$$2\int_0^{\frac{\pi}{2}} \mathrm{d\theta} \;= 2\;\; \theta]_0^{\frac{\pi}{2}} = \pi$$

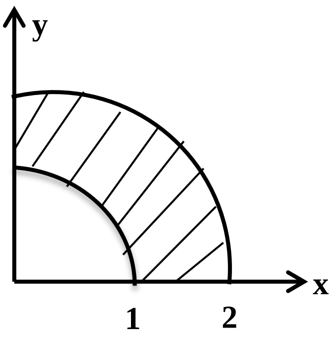
Example 7: $\iint_R \frac{1}{x^2+y^2}\,dA$, bounded (محددة) by $\;x^2 + y^2 = 1 \;\;$ &

$x^2 + y^2 = 4$ in the first quardant (الربع الاول)

Solution:

$$\int_0^{\frac{\pi}{2}} \int_1^2 \frac{1}{r^2}\,\mathrm{r\,dr\,d\theta} = \int_0^{\frac{\pi}{2}} \int_1^2 \frac{1}{r}\,\mathrm{dr\,d\theta} = \int_0^{\frac{\pi}{2}} Lnr]_1^2 \mathrm{d\theta} = \int_0^{\frac{\pi}{2}} [Ln2 - Ln1]\;\mathrm{d\theta} =$$

$$\int_0^{\frac{\pi}{2}} Ln2\;\;\mathrm{d\theta} = Ln2\;\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{2}\;Ln2$$

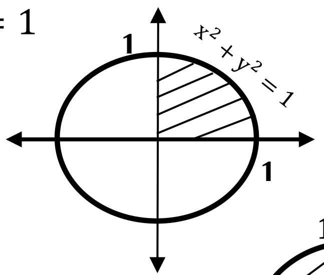


Or

$$\int_1^2 \int_0^{\frac{\pi}{2}} \frac{1}{r^2}\;\mathrm{r\,d\theta\,dr} \;= \int_1^2 \int_0^{\frac{\pi}{2}} \frac{1}{r}\;\mathrm{d\theta dr} \;= \int_1^2 \frac{1}{r}\;\theta\Big]_0^{\frac{\pi}{2}} \mathrm{dr}$$

$$\int_1^2 \frac{\pi}{2}\frac{1}{r}\;\mathrm{dr} = \frac{\pi}{2}\;\mathrm{Ln\;r}]_1^2 = \frac{\pi}{2}\;\mathrm{Ln\;2}$$

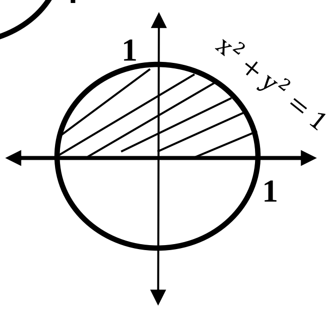
Example 8: Evaluate the double integral by using the polar Coordinates & Cartesian Coordinates for $\;\mathrm{x}^2 + \mathrm{y}^2 = 1$



Solution: In Polar Coordinates:

$$A = \mathbf{4}\int_0^1 \int_0^{\frac{\pi}{2}} r\;\mathrm{d\theta\;dr} \;= \mathbf{4}\int_0^1 r\;\theta]_0^{\frac{\pi}{2}}\mathrm{dr}$$

$$= 2\pi\int_0^1 r\;\mathrm{dr} = 2\pi\frac{r^2}{2}\Big]_0^1 = \pi$$



or

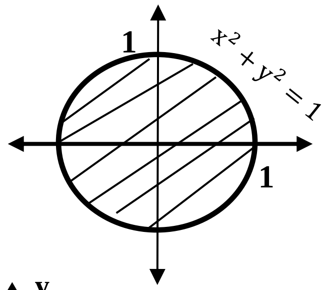
$$A = \mathbf{2}\int_0^1 \int_0^{\pi} r\;\mathrm{d\theta\;dr} \;= \mathbf{2}\int_0^1 r\;\theta]_0^{\pi}\mathrm{dr}$$

$$= 2\pi\int_0^1 r\;\mathrm{dr} = 2\pi\frac{r^2}{2}\Big]_0^1 = \pi$$

or

$$A = \int_0^1 \int_0^{2\pi} r \, d\theta \, dr = \int_0^1 r \, \theta]_0^{2\pi} dr$$

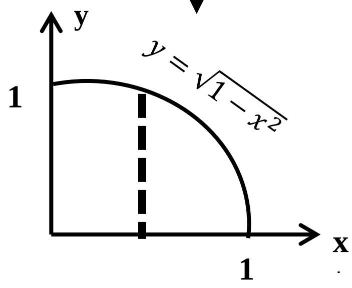
$$= 2\pi \int_0^1 r \, dr = 2\pi \frac{r^2}{2}\Big]_0^1 = \pi$$



In Cartesian Coordinates

$$x^2 + y^2 = 1 \quad \Rightarrow \quad y^2 = 1 - x^2 \Rightarrow \quad y = \sqrt{1-x^2}$$

$$4\int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx = 4\int_0^1 \sqrt{1-x^2}\; dx$$



Let $x = sin\theta \quad \Rightarrow \quad dx = cos\theta \, d\theta,$

Substitute in above integral we have

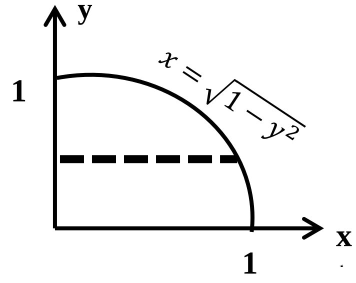
$$4\int_0^1 \sqrt{1-sin^2\theta}\; cos\theta \, d\theta = 4\int_0^1 \sqrt{cos^2\theta}\; cos\theta \, d\theta = 4\int_0^1 cos^2\theta \, d\theta$$

$$4\int_0^1 \frac{1+cos2\theta}{2} d\theta = 4\int_0^1 \left[\frac{1}{2} + \frac{cos2\theta}{2}\right] d\theta = 2\theta]_0^1 + sin2\theta]_0^1$$

$$= 2 + sin2$$

or

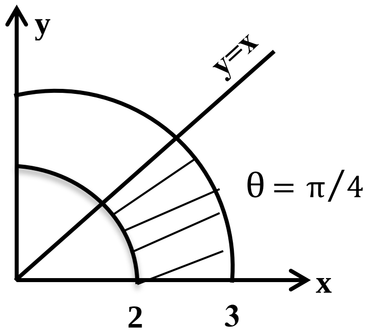
$$4\int_0^1 \int_0^{\sqrt{1-y^2}} dxdy = 4\int_0^1 \sqrt{1-y^2}\; dx$$



Let $y = sin\theta \quad \Rightarrow \quad dy = cos\theta \, d\theta,$

Example 9: Use polar coordinates to evaluate $\iint_R \frac{y^2}{x^2}\, dA$, where R is that part of annulus(الحلقة) $4 \le x^2 + y^2 \le 9$ lying in the first quadrant and below the line **y=x**.

Solution: $\iint_R \frac{y^2}{x^2}\, dA = \int_0^{\frac{\pi}{4}} \int_2^3 \frac{r^2 sin^2\theta}{r^2 cos^2\theta}\; r dr\, d\theta$



$$\int_0^{\frac{\pi}{4}}\int_2^3 tan^2\theta \; r\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \frac{r^2}{2}\Big]_2^3 \; tan^2\theta \; d\theta$$

$$\frac{5}{2}\int_0^{\frac{\pi}{4}}[sec^2\theta - 1]\,d\theta = \frac{5}{2}[tan\theta - \theta]_0^{\frac{\pi}{4}} = \frac{5}{2}\left[\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right]$$

Or

$$\int_2^3\int_0^{\frac{\pi}{4}} tan^2\theta \; r\,d\theta dr = \int_2^3\int_0^{\frac{\pi}{4}} r \; tan^2\theta\,d\theta\,dr = \int_2^3\int_0^{\frac{\pi}{4}} r[\,sec^2\theta - 1]\,d\theta\,dr$$

$$\int_2^3 r[tan\theta - \theta]_0^{\frac{\pi}{4}}\,dr = \int_2^3\left[\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right]rdr = \left[\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right] \; \frac{r^2}{2}\Big|_2^3 = \frac{5}{2}\left[\frac{1}{\sqrt{2}} - \frac{\pi}{4}\right] .$$

Example 10: $\int_0^1\int_y^{\sqrt{2-y^2}} x \; dx\,dy$

Solution: $0 \le y \le 1$, $x = y$, $x = \sqrt{2-y^2}$ $\Rrightarrow$ $x^2 + y^2 = 2$

$$\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{2}} r^2 \; cos\theta\,dr\,d\theta = \int_0^{\frac{\pi}{4}} cos\theta \; \frac{r^3}{3}\Big]_0^{\sqrt{2}} \; d\theta$$

$$\frac{2\sqrt{2}}{3}\int_0^{\frac{\pi}{4}} cos\theta\,d\theta = \frac{2\sqrt{2}}{3}[sin\theta]_0^{\frac{\pi}{4}} = \frac{2\sqrt{2}}{3}\left(\frac{1}{\sqrt{2}}\right) = \frac{2}{3}$$