11th Lecture

Chapter Four

التراص في الفضاءات التبولوجية

Compactness in Topological Spaces

Definition (4.1): Let E be a subset in a topological space (X, τ)

(i) We say that the family $\{G_\alpha\}_{\alpha\in\Lambda}$ is a **cover** (غطاء) for E iff $\subset \bigcup_{\alpha\in\Lambda} G_\alpha$.

(ii) We say that the subfamily of $\{G_\alpha\}_{\alpha\in\Lambda}$ (say $\{G_{\alpha_i}\}_{\alpha_i\in\Lambda}$ is a **subcover** (غطاء جزئي) of E) iff

$E \subset \bigcup_{\alpha_i\in\Lambda} G_{\alpha_i}$.

(iii) We say that the family $\{G_\alpha\}_{\alpha\in\Lambda}$ is an **open cover** (غطاء مفتوح) of E iff $E \subset \bigcup_{\alpha\in\Lambda} G_\alpha$ and

$G_\alpha \in \tau, \forall\ \alpha \in \Lambda$.

(iv) We say that $\{G_{\alpha_i}\}_{i=1}^{n}$ is an **open finite subcover** (غطاء جزئي منتهي مفتوح) of E iff $E \subset \bigcup_{i=1}^{n} G_{\alpha_i}$, where

$G_{\alpha_i} \in \tau$.

Definition (4.2): If E is a subset in (X, τ). We say that E is **compact** (متراصة) iff for every open cover $\{G_\alpha\}_{\alpha\in\Lambda}$ of E, there exists a finite subcover of E (say $\{G_{\alpha_i}\}_{i=1}^{n}$).

In other word: (E compact) $\Leftrightarrow \left[\left(E \subset \bigcup_{\alpha\in\Lambda} G_\alpha\right) \Rightarrow \left(E \subset \bigcup_{i=1}^{n} G_{\alpha_i}\right)\right]$

Example (4.1): If $E = \{x_1, x_2, x_3, \dots, x_n\}$ is finite (X, τ). Then E compact.

Proof: Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be an open cover of E

$\Rightarrow\ E \subset \bigcup_{\alpha\in\Lambda} G_\alpha \Rightarrow\ \exists\ G_{\alpha_i}, x_i \in G_{\alpha_i}, i = 1,2, \dots, n$

$\Rightarrow\ x_1 \in G_{\alpha_1}, x_2 \in G_{\alpha_2}, \dots x_n \in G_{\alpha_n}$

$\Rightarrow\ \{x_1\} \subset G_{\alpha_1}, \{x_2\} \subset G_{\alpha_2}, \dots \{x_n\} \subset G_{\alpha_n}$

$\Rightarrow\ \bigcup_{i=1}^{n}\{x_i\} \subset \bigcup_{i=1}^{n} G_{\alpha_i}$

$\Rightarrow\ E \subset \bigcup_{i=1}^{n} G_{\alpha_i}$

$\Rightarrow\ \{G_{\alpha_i}\}_{i=1}^{n}$ is a finite subcover of E.

$\Rightarrow\ E$ is compact.

Example (4.2): Determine whether $E = (0,1)$ is compact or not in (R, τ).

Proof: Let $\{G_n\}_{n\in N}$ such that $G_n = (\frac{1}{n+1}, 1)$ be an open cover of .

$\Rightarrow\ E \subset \bigcup_{\alpha\in\Lambda} (\frac{1}{n+1}, 1\,)$

Let (a_1, b_1), (a_2, b_2), $\ldots$, (a_n, b_n) be a finite subfamily of $\left\{(\frac{1}{n+1}, 1)\right\}_{n\in N}$

Put $\varepsilon = min(a_1, a_2, \ldots, a_m)$

$\Rightarrow\ \bigcup_{i=1}^{m}(a_i, b_i) = (\varepsilon, b_m)$

Let $b_m = 1$

$\Rightarrow\ \bigcup_{i=1}^{m}(a_i, b_i) = (\varepsilon, 1)$

Now $(0,1) \not\subset (\varepsilon, 1)$,   for $\varepsilon > 0$

$\Rightarrow\ (0,1) \not\subset \bigcup_{i=1}^{m}(a_i, b_i)$

There is no finite subcovering for E

$\Rightarrow\ E$ is not compact.

Theorem (4.1): Let (X^*, τ^*) be a topological subspace of (X, τ) and $E \subset X^*$. Then E is τ^*-compact iff E is τ-compact.

Proof: Suppose that E is τ^*-compact

We need to prove that E is τ-compact

Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be an open cover of E

$\Rightarrow\ E \subset \bigcup_{\alpha\in\Lambda} G_\alpha$

$\Rightarrow\ E \cap X^* \subset (\bigcup_{\alpha\in\Lambda} G_\alpha) \cap X^*$

$\Rightarrow\ E \cap X^* \subset \bigcup_{\alpha\in\Lambda}(G_\alpha \cap X^*)$

$\Rightarrow\ E \subset \bigcup_{\alpha\in\Lambda} G_\alpha^* \quad \Rightarrow \quad \{G_\alpha^*\}_{\alpha\in\Lambda}$ is an open cover for E

But E is τ^*-compact

$\Rightarrow\ \exists\ \{G_{\alpha_i}^*\}_{i=1}^{n}$ finite subcover for E

$\Rightarrow\ E \subset \bigcup_{i=1}^{n} G_{\alpha_i}^*$

$\Rightarrow\ E \subset \bigcup_{i=1}^{n}\left(G_{\alpha_i} \cap X^*\right)$

$\Rightarrow\ E \subset (\bigcup_{i=1}^{n} G_{\alpha_i}) \cap X^*$

$\Rightarrow\ E \subset (\bigcup_{i=1}^{n} G_{\alpha_i})$

$\therefore\ \left(E \subset \bigcup_{\alpha\in\Lambda} G_\alpha\right)\ \Rightarrow\ \left(E \subset \bigcup_{i=1}^{n} G_{\alpha_i}\right)$

$\Rightarrow\ E$ is τ-compact

Conversely: Suppose ($\Leftarrow$) E is τ –compact

We need to prove that E is τ^*-compact

Let $\{G_\alpha^*\}_{\alpha\in\Lambda}$ be an open cover for E

$\Rightarrow\ E \subset \bigcup_{\alpha\in\Lambda} G_\alpha^*$

$\Rightarrow\ E \subset \bigcup_{\alpha\in\Lambda}(G_\alpha \cap X^*)$

$\Rightarrow\ E \subset (\bigcup_{\alpha\in\Lambda} G_\alpha) \cap X^* \subset (\bigcup_{\alpha\in\Lambda} G_\alpha)$

$\Rightarrow\ E \subset \bigcup_{\alpha\in\Lambda} G_\alpha$

But E is τ-compact

$\Rightarrow\ E \subset \bigcup_{i=1}^{m} G_{\alpha_i}$

$\Rightarrow\ E \cap X^* \subset (\bigcup_{i=1}^{m} G_{\alpha_i}) \cap X^*$

$\Rightarrow\ E \subset \bigcup_{i=1}^{m} G_{\alpha_i}^*$

$\therefore\ (E \subset \bigcup_{\alpha\in\Lambda} G_\alpha^*)\ \Rightarrow\ (E \subset \bigcup_{i=1}^{m} G_{\alpha_i}^*)\ \Rightarrow\ E$ is τ^*- compact.