Example 7. *Find the Fourier transform of the function*

$$f(u) = \begin{cases} u^2, & \text{if } |u| < u_0 \\ 0, & \text{if } |u| > u_0 \end{cases}.$$

Sol. We have $$f(u) = \begin{cases} u^2, & \text{if } -u_0 < u < u_0 \\ 0, & \text{if } u < -u_0 \text{ or } u > u_0 \end{cases}$$

$$\begin{aligned}
F(f(u)) &= \int_{-\infty}^{\infty} f(u)\, e^{-isu}\, du \\
&= \int_{-\infty}^{-u_0} f(u)\, e^{-isu}\, du + \int_{-u_0}^{u_0} f(u)\, e^{-isu}\, du + \int_{u_0}^{\infty} f(u)\, e^{-isu}\, du \\
&= \int_{-\infty}^{-u_0} 0\,.\, e^{-isu}\, du + \int_{-u_0}^{u_0} u^2\, e^{-isu}\, du + \int_{u_0}^{\infty} 0\,.\, e^{-isu}\, du \\
&= 0 + \left(u^2 \,.\, \frac{e^{-isu}}{-is} - 2u \,.\, \frac{e^{-isu}}{(-is)^2} + 2 \,.\, \frac{e^{-isu}}{(-is)^3} \right)\Bigg|_{-u_0}^{u_0} + 0 \\
&= -\frac{1}{is}(u_0^2\, e^{-isu_0} - u_0^2\, e^{isu_0}) + \frac{2}{s^2}(u_0 e^{-isu_0} + u_0 e^{isu_0}) + \frac{2}{is^3}(e^{-isu_0} - e^{isu_0}) \\
&= \frac{2u_0^2}{s}\left(\frac{e^{isu_0} - e^{-isu_0}}{2i}\right) + \frac{4u_0}{s^2}\left(\frac{e^{isu_0} + e^{-isu_0}}{2}\right) - \frac{4}{s^3}\left(\frac{e^{isu_0} - e^{-isu_0}}{2i}\right) \\
&= \frac{2u_0^2}{s} \sin su_0 + \frac{4u_0}{s^2} \cos su_0 - \frac{4}{s^3} \sin su_0 \\
&= \mathbf{\frac{2}{s^3}[(u_0^2 s^2 - 2) \sin su_0 + 2u_0 s \cos su_0].}
\end{aligned}$$

Example 8. *Find the Fourier transform of* $f(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$ *and hence evaluate :*

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin sa \cos sx}{s}\, ds$ *and* *(ii)* $\displaystyle\int_{0}^{\infty} \frac{\sin s}{s}\, ds.$

Sol. We have $$f(x) = \begin{cases} 1, & \text{if } -a < x < a \\ 0, & \text{if } x < -a \text{ or } x > -a \end{cases}$$

$$\begin{aligned}
F(f(x)) &= \int_{-\infty}^{\infty} f(x) e^{-isx}\, dx \\
&= \int_{-\infty}^{-a} f(x) e^{-isx}\, dx + \int_{-a}^{a} f(x) e^{-isx}\, dx + \int_{a}^{\infty} f(x) e^{-isx}\, dx \\
&= \int_{-\infty}^{-a} 0\,.\, e^{-isx}\, dx + \int_{-a}^{a} 1\,.\, e^{-isx}\, dx + \int_{a}^{\infty} 0\,.\, e^{-isx}\, dx \\
&= 0 + \frac{e^{-isx}}{-is}\Bigg|_{-a}^{a} + 0 = -\frac{1}{is}(e^{-isa} - e^{isa}) \\
&= \frac{2}{s}\left(\frac{e^{isa} - e^{-isa}}{2i}\right) = \frac{2}{s} \sin sa
\end{aligned}$$

$\therefore$ Fourier transform of $f(x)$

$$= \bar{f}(s) = F(f(x)) = \frac{\mathbf{2 \sin as}}{\mathbf{s}}.$$

(*i*) By inverse Fourier transform, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)e^{isx}\, ds.$$

$$\Rightarrow \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin as}{s}(\cos sx + i\sin sx)\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin as\cos sx}{s} + i\,\frac{\sin as\sin sx}{s}\right) ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin as\cos sx}{s}\, ds + \frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\sin as\sin sx}{s}\, ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin as\cos sx}{s}\, ds + \frac{i}{\pi}\,.\,0$$

$$\left(\because \quad \frac{\sin as\sin sx}{s} \text{ is an odd function of } s\right)$$

$$\therefore \qquad f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin as\cos sx}{s}\, ds$$

$$\therefore \qquad \int_{-\infty}^{\infty}\frac{\sin as\cos sx}{s}\, ds = \pi f(x) = \begin{cases}\pi, & \textbf{if } |\mathbf{x}| < \mathbf{a} \\ \mathbf{0}, & \textbf{if } |\mathbf{x}| > \mathbf{a}\end{cases}.$$

(*ii*) We have $\displaystyle\int_{-\infty}^{\infty}\frac{\sin as\cos sx}{s}\, ds = \begin{cases}\pi, & \text{if } |x| < a \\ 0, & \text{if } |x| > a\end{cases}$

In particular let $x = 0$ and $a = 1$.

$$\therefore \qquad \int_{-\infty}^{\infty}\frac{\sin(1\,.\,s)\cos(s\,.\,0)}{s}\, ds = \pi \qquad (\because \quad |x| = |0| = 0 \quad \text{and} \quad a = 1)$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty}\frac{\sin s}{s}\, ds = \pi \quad \Rightarrow \quad 2\int_{0}^{\infty}\frac{\sin s}{s}\, ds = \pi$$

$$\therefore \qquad \int_{0}^{\infty}\frac{\sin s}{s}\, ds = \frac{\pi}{\mathbf{2}}.$$

Example 9. *Find the Fourier transform of the function*

$$f(x) = \begin{cases}1 - x^2, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1\end{cases}$$

and hence evaluate $\displaystyle\int_0^{\infty}\frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\, dx.$

Sol. We have $\qquad f(x) = \begin{cases}1 - x^2, & \text{if } -1 < x < 1 \\ 0, & \text{if } x < -1 \text{ or } x > 1\end{cases}.$

$$F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{-1} f(x)\, e^{-isx}\, dx + \int_{-1}^{1} f(x)\, e^{-isx}\, dx + \int_{1}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{-1} 0 \,.\, e^{-isx}\, dx + \int_{-1}^{1} (1-x^2)\, e^{-isx}\, dx + \int_{1}^{\infty} 0 \,.\, e^{-isx}\, dx$$

$$= 0 + \left[(1-x^2) \,.\, \frac{e^{-isx}}{-is} - (-2x) \,.\, \frac{e^{-isx}}{(-is)^2} + (-2) \,.\, \frac{e^{-isx}}{(-is)^3} \right]_{-1}^{1} + 0$$

(By using general rule of integration by parts)

$$= 0 - 0 + \left[-\frac{2x}{s^2} e^{-isx} + \frac{2i}{s^3} e^{-isx} \right]_{-1}^{1}$$

$$= -\frac{2}{s^2}\,(e^{-is} + e^{is}) + \frac{2i}{s^3}\,(e^{-is} - e^{is})$$

$$= -\frac{4}{s^2}\left(\frac{e^{is} + e^{-is}}{2}\right) - \frac{4i^2}{s^3}\left(\frac{e^{is} - e^{-is}}{2i}\right)$$

$$= -\frac{4}{s^2}\cos s + \frac{4}{s^3}\sin s = \frac{4}{s^3}\,(\sin s - s\cos s)$$

$\therefore$ Fourier transform of $f(x)$

$$= \bar{f}(s) = \mathrm{F}(f(x)) = \mathbf{\frac{4}{s^3}\,(\sin s - s\cos s).}$$

By Inverse Fourier transform, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx}\, ds$$

$$\Rightarrow \qquad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{s^3}(\sin s - s\cos s)\, e^{isx}\, ds = f(x)$$

$$\Rightarrow \qquad \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{1}{s^3}(\sin s - s\cos s)(\cos sx + i\sin sx)\, ds = \begin{cases} 1-x^2, & \text{if } |x|<1 \\ 0, & \text{if } |x|>1 \end{cases}$$

$$\Rightarrow \qquad \int_{-\infty}^{\infty} \left(\frac{(\sin s - s\cos s)\cos sx}{s^3} + i\,\frac{(\sin s - s\cos s)\sin sx}{s^3} \right) ds = \begin{cases} \frac{\pi}{2}(1-x^2), & \text{if } |x|<1 \\ 0, & \text{if } |x|>1 \end{cases}$$

Equating the real parts, we have

$$\int_{-\infty}^{\infty} \frac{(\sin s - s\cos s)\cos sx}{s^3}\, ds = \begin{cases} \frac{\pi}{2}(1-x^2), & \text{if } |x|<1 \\ 0, & \text{if } |x|>1 \end{cases}$$

$$\Rightarrow \qquad 2\int_{0}^{\infty} \frac{(\sin s - s\cos s)\cos sx}{s^3}\, ds = \begin{cases} \frac{\pi}{2}(1-x^2), & \text{if } |x|<1 \\ 0, & \text{if } |x|>1 \end{cases}$$

($\because$ The integrand is an even function of s)

$$\Rightarrow \qquad \int_{0}^{\infty} \frac{(s\cos s - \sin s)\cos sx}{s^3}\, ds = \begin{cases} \frac{\pi}{4}(x^2-1), & \text{if } |x|<1 \\ 0, & \text{if } |x|>1 \end{cases}$$

Taking $x = \frac{1}{2}$, we get

$$\int_0^\infty \frac{(s\cos s - \sin s)\cos s/2}{s^3}\,ds = \frac{\pi}{4}\left(\left(\frac{1}{2}\right)^2 - 1\right) = -\frac{3\pi}{16} \qquad \left(\because \left|\frac{1}{2}\right| < 1\right)$$

$\therefore$ By changing the variable of integration, we get

$$\int_0^\infty \frac{x\cos x - \sin x}{x^3}\cos\frac{x}{2}\,dx = -\frac{\mathbf{3\pi}}{\mathbf{16}}.$$

Remark. In the above question, instead of comparing real and imaginary parts, we can also use the fact that $\dfrac{(\sin s - s\cos s)\sin sx}{s^3}$ is an odd function of s.

TEST YOUR KNOWLEDGE

Find the Fourier transform of the following functions (Q. 1—10) :

1. $f(x) = \begin{cases} 4, & \text{if } 0 < x < 2 \\ 0, & \text{otherwise} \end{cases}$

2. $f(x) = \begin{cases} 1, & \text{if } 3 < x < 9 \\ 0, & \text{otherwise} \end{cases}$

3. $f(x) = \begin{cases} 2x, & 0 < x < 7 \\ 0, & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} x, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$

5. $f(x) = \begin{cases} -1, & \text{if } -4 < x < 0 \\ 1, & \text{if } 0 < x < 4 \\ 0, & \text{otherwise} \end{cases}$

6. $f(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$

7. $f(x) = \begin{cases} \dfrac{\sqrt{2\pi}}{2l}, & \text{if } |x| < l \\ 0, & \text{if } |x| > l \end{cases}$

8. $f(x) = \begin{cases} \dfrac{1}{2\varepsilon}, & \text{if } |x| \le \varepsilon \\ 0, & \text{if } |x| > \varepsilon \end{cases}$

9. $f(x) = e^{-|x|}$

10. $f(x) = e^{-a|x|}$, $a > 0$.

11. Find the Fourier transform of $f(x) = \begin{cases} 1, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$. Hence evaluate $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

Answers

1. $\dfrac{4i}{s}(e^{-2is} - 1)$

2. $\dfrac{i}{s}(e^{-9is} - e^{-3is})$

3. $\dfrac{2(e^{-7is}(1 + 7is) - 1)}{s^2}$

4. $\dfrac{e^{-ias}(1 + ias) - 1}{s^2}$

5. $\dfrac{2i}{s}(\cos s - 1)$

6. $\dfrac{2\sin as}{s}$

7. $\dfrac{\sqrt{2\pi}}{ls}\sin ls$

8. $\dfrac{\sin \varepsilon s}{\varepsilon s}$

9. $\dfrac{2}{1 + s^2}$

10. $\dfrac{2a}{a^2 + s^2}$

11. $\dfrac{2\sin s}{s}, \dfrac{\pi}{2}$.

5.7. FOURIER SINE AND COSINE TRANSFORMS

Let f be a real valued function of the real variable x such that $f(x)$ and $f'(x)$ are piecewise continuous in every finite interval and the integral of $|f(x)|$ exists from $-\infty$ to ∞. The integral

$\int_0^\infty f(x) \sin sx\, dx$ is called the **Fourier sine transform** of the function $f(x)$ and we write this as $F_S(f(x))$ or as $\bar{f}_S(s)$. Thus, $\mathbf{\bar{f}_S(s) = F_S(f(x)) = \int_0^\infty f(x) \sin sx\, dx}$.

Similarly, the integral $\int_0^\infty f(x) \cos sx\, dx$ is called the **Fourier cosine transform** of the function $f(x)$ and we write this as $F_C(f(x))$ or as $\bar{f}_C(s)$.

Thus, $\mathbf{\bar{f}_C(s) = F_C(f(x)) = \int_0^\infty f(x) \cos sx\, dx}$.

By Fourier integral expression of $f(x)$, we have

$$f(x) = \int_0^\infty (A(s) \cos sx + B(s) \sin sx)\, ds, \qquad \text{...(1)}$$

where $$A(s) = \frac{1}{\pi}\int_{-\infty}^\infty f(v) \cos sv\, dv \quad \text{and} \quad B(s) = \frac{1}{\pi}\int_{-\infty}^\infty f(v) \sin sv\, dv.$$

Case I. f(x) is an odd function.

$\therefore$ $f(v) \cos sv$ is an odd function of v.

$$\therefore \quad \int_{-\infty}^\infty f(v) \cos sv\, dv = 0$$

$$\therefore \quad A(s) = \frac{1}{\pi} \,.\, 0 = 0$$

Also, $f(v)$ $\sin sv$ is an even function of v.

$$\therefore \quad \int_{-\infty}^\infty f(v) \sin sv\, dv = 2\int_0^\infty f(v) \sin sv\, dv$$

$$\therefore \quad B(s) = \frac{2}{\pi}\int_0^\infty f(v) \sin sv\, dv$$

$$\therefore \ (1) \Rightarrow \quad f(x) = \int_0^\infty \left(0 \,.\, \cos sx + \left(\frac{2}{\pi}\int_0^\infty f(v) \sin sv\, dv\right) \sin sx\right) ds$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty f(v) \sin sv\, dv\right) \sin sx\, ds$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi}\int_0^\infty \bar{f}_S(s) \sin sx\, ds.$$

The function $\frac{2}{\pi}\int_0^\infty \bar{f}_S(s) \sin sx\, ds$ *i.e.*, $f(x)$ is called the **inverse Fourier sine transform** of the function $\bar{f}_S(s)$ and we write $F_S^{-1}(\bar{f}_S(s)) = F_S^{-1}(F_S(f(x)) = f(x)$.

Case II. f(x) is an even function.

$\therefore$ $f(v) \cos sv$ is an even function of v.

$$\therefore \quad \int_{-\infty}^\infty f(v) \cos sv\, dv = 2\int_0^\infty f(v) \cos sv\, dv$$

$$\therefore \quad A(s) = \frac{2}{\pi}\int_0^\infty f(v) \cos sv\, dv$$

Also, $f(v)$ sin sv is an odd function of v.

$$\therefore \quad \int_{-\infty}^{\infty} f(v) \sin sv \, dv = 0$$

$$\therefore \quad B(s) = \frac{1}{\pi} \cdot 0 = 0$$

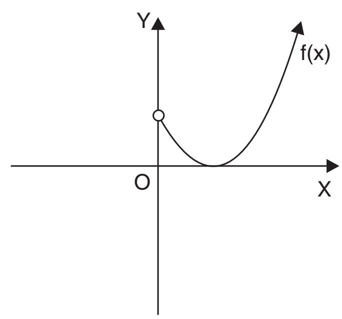
$$\therefore \quad (1) \Rightarrow \quad f(x) = \int_0^{\infty} \left(\left(\frac{2}{\pi} \int_0^{\infty} f(v) \cos sv \, dv \right) \cos sx + 0 \cdot \sin sx \right) ds$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(v) \cos sv \, dv \right) \cos sx \, ds$$

$$\Rightarrow \quad f(x) = \frac{2}{\pi} \int_0^{\infty} \bar{f}_C(s) \cos sx \, ds.$$

The function $\frac{2}{\pi} \int_0^{\infty} \bar{f}_C(s) \cos sx \, ds$ *i.e.*, $f(x)$ is called the **inverse Fourier cosine transform** of the function $\bar{f}_C(s)$ and we write

$$F_C^{-1}(\bar{f}_C(s)) = F_C^{-1}(F_C(f(x)) = f(x).$$



Important Note. If a function is defined on the interval $(0, \infty)$ then it can be defined on the interval $(-\infty, 0)$ so that the extension of the function under consideration is an even (resp. odd) function. Let $f(x)$ be a function defined on $(0, \infty)$ with graph as shown in the figure. The function shown in the figure I and figure II are respectively the even extension of $f(x)$ and the odd extension of $f(x)$.

$\therefore$ Any function defined on $(0, \infty)$ can be considered to be an even function as well as an odd function.

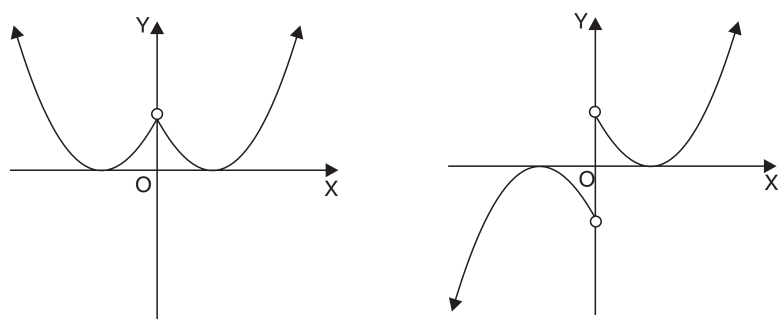


Figure I. Even extension of f(x) Figure II. Odd extension of f(x)

$\therefore$ If a function is defined on $(0, \infty)$ then the formulae regarding inverse Fourier sine transform and inverse Fourier cosine transform are applicable.

Remark. We know that

$$\int e^{ax} \sin (bx + c) \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin (bx + c)) - (b \cos (bx + c)) + C$$

and
$$\int e^{ax} \cos (bx + c) \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos (bx + c) + b \sin (bx + c)) + C.$$

$$\therefore \quad \int_0^\infty e^{-ax} \sin bx \, dx = \frac{e^{-ax}}{a^2+b^2}(-a \sin bx - b \cos bx)\Bigg|_0^\infty$$

$$= \lim_{x \to \infty} \frac{1}{e^{ax}} \cdot \frac{-(a \sin bx + b \cos bx)}{a^2+b^2} - \frac{1}{a^2+b^2}(-a \,.\, 0 - b \,.\, 1)$$

$$= 0 + \frac{b}{a^2+b^2} = \mathbf{\frac{b}{a^2+b^2}}.$$

Similarly, $\displaystyle\int_0^\infty e^{-ax} \cos bx \, dx = \mathbf{\frac{a}{a^2+b^2}}.$

WORKING RULES FOR SOLVING PROBLEMS

Rule I. (*i*) *The Fourier sine transform of the function f(x) is given by*

$$\bar{f}_S(s) = F_S(f(x)) = \int_0^\infty f(x) \sin sx \, dx.$$

(*ii*) *The Fourier cosine transform of the function f(x) is given by*

$$\bar{f}_C(s) = F_C(f(x)) = \int_0^\infty f(x) \cos sx \, dx.$$

Rule II. (*i*) *The inverse Fourier sine transform of the function g(s) is given by*

$$F_S^{-1}(g(s)) = \frac{2}{\pi}\int_0^\infty g(s) \sin sx \, ds.$$

(*ii*) *The inverse Fourier cosine transform of the function g(s) is given by*

$$F_C^{-1}(g(s)) = \frac{2}{\pi}\int_0^\infty g(s) \cos sx \, ds.$$

Rule III. (*i*) *If f(x) is either an odd function of x or defined on (0, ∞), then*

$$F_S^{-1}(\bar{f}_S(s)) = f(x).$$

(*ii*) *If f(x) is either an even function of x or defined on (0, ∞), then*

$$F_C^{-1}(\bar{f}_C(s)) = f(x).$$

TYPE I. Problems Based on Fourier Sine and Cosine Transforms

ILLUSTRATIVE EXAMPLES

Example 1. *Find the Fourier sine and cosine transforms of the function e^{-ax}, $a > 0$.*

Sol. Let $f(x) = e^{-ax}$, $a > 0$.

$$\therefore \quad F_S(f(x)) = \int_0^\infty f(x) \sin sx \, dx$$

$$= \int_0^\infty e^{-ax} \sin sx \, dx = \mathbf{\frac{s}{a^2+s^2}}. \qquad \left(\text{Using } \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}\right)$$

Also, $\displaystyle F_C(f(x)) = \int_0^\infty f(x) \cos sx \, dx$

$$= \int_0^\infty e^{-ax} \cos sx \, dx = \mathbf{\frac{a}{a^2+s^2}}. \qquad \left(\text{Using } \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2+b^2}\right)$$