proposition

Let $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$ be two convergent sequence in normed space $(X, \|\cdot\|)$ such that

$$x_n \longrightarrow x_0$$
$$y_n \longrightarrow y_0 \qquad x_0, y_0 \in X$$

Then

① $(x_n + y_n) \longrightarrow x_0 + y_0$

② $\alpha x_n \longrightarrow \alpha x_0 \qquad \alpha \in K$

③ $\|x_n\| \longrightarrow \|x_0\|$

④ $\|x_n - y_n\| \longrightarrow \|x_0 - y_0\|$

proof. ① $\because x_n \longrightarrow x_0$, for $\frac{\epsilon}{2} > 0$, there exist $k_1 \in N$ such that

$$\|x_n - x_0\| < \frac{\epsilon}{2} \qquad \forall n > k_1$$

$\because y_n \longrightarrow y_0$, for $\frac{\epsilon}{2} > 0$, there exist $k_2 \in N$ such that

$$\|y_n - y_0\| < \frac{\epsilon}{2} \qquad \forall n > k_2 .$$

Let $k = \max\{k_1, k_2\}$

$$\|(x_n + y_n) - (x_0 - y_0)\| = \|(x_n - x_0) + (y_n - y_0)\|$$
$$\leq \|x_n - x_0\| + \|y_n - y_0\|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \forall n > k$$

$\therefore (x_n + y_n) \longrightarrow x_0 + y_0$

(2) $\because x_n \to x_0$, for $\frac{\epsilon}{|\alpha|} > 0$ where $\alpha \in K$ field

there exist $k \in N$ such that

$\| x_n - x_0 \| < \frac{\epsilon}{|\alpha|} \quad \forall n > k$

$\| \alpha x_n - \alpha x_0 \| = \| \alpha \cdot (x_n - x_0) \|$

$= |\alpha| \, \| x_n - x_0 \|$

$< \cancel{|\alpha|} \, \frac{\epsilon}{\cancel{|\alpha|}} \quad \forall n > k$

$= \epsilon$

$\therefore \| \alpha x_n - \alpha x_0 \| < \epsilon \quad \forall n > k$

$\therefore \alpha x_n \longrightarrow \alpha \cdot x_0$

(3) $\because x_n \to x$, for $\epsilon > 0$, $\exists k \in N$ such that

$\| x_n - x_0 \| < \epsilon \quad \forall n > k$

$\because \big| \| x_n \| - \| x_0 \| \big| < \| x_n - x_0 \| < \epsilon \quad \forall n > k$

$\therefore \| x_n \| \longrightarrow \| x_0 \|$

(4) $\because x_n \longrightarrow x_0$, for $\frac{\epsilon}{2} > 0$, $\exists k_1 \in N$ such that

$\| x_n - x_0 \| < \frac{\epsilon}{2} \quad \forall n > k_1$

$\because y_n \longrightarrow y_0$, for $\frac{\epsilon}{2} > 0$, $\exists k_2 \in N$ such that

$\| y_n - y_0 \| < \frac{\epsilon}{2} \quad \forall n > k_2$

Let $K = \max\{k_1, k_2\}$

$$\left| \|x_n - y_n\| - \|x_0 - y_0\| \right| \leq \|x_n - y_n - (x_0 - y_0)\|$$

$$= \|(x_n - x_0) + (y_0 - y_n)\|$$

$$\leq \|(x_n - x_0\| + \|y_0 - y_n\|$$

$$= \|x_n - x_0\| + \|y_n - y_0\|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall n > k$$

$$\therefore \|x_n - y_n\| \to \|x_0 - y_0\|$$

Example Let R^3 be a normed space over R

Let $x \in R^3$, $x = (x_1, x_2, x_3)$

$$\|x\| = |x_1| + |x_2| + |x_3|$$

Let $x_n = (0, 1 + \frac{1}{n}, \frac{1}{n})$

$y_n = (1, -\frac{1}{n}, 2)$

find its ① $(x_n)_{n=1}^{\infty}$ ② $(y_n)_{n=1}^{\infty}$, ③ $(x_n + y_n)_{n=1}^{\infty}$

④ $(x_n - y_n)_{n=1}^{\infty}$, ⑤ $(3x_n)_{n=1}^{\infty}$

⑥ $(\|x_n\|)_{n=1}^{\infty}$, ⑦ $(\|x_n - y_n\|)_{n=1}^{\infty}$

convergent sequence.

(1) $x_n \longrightarrow x_0 = (0, 1, 0)$

$$\| x_n - x_0 \| = \| (0, 1+\tfrac{1}{n}, \tfrac{1}{n}) - (0, 1, 0) \|$$
$$= \| (0, \tfrac{1}{n}, \tfrac{1}{n}) \|$$
$$= |0| + |\tfrac{1}{n}| + |\tfrac{1}{n}|$$
$$= \tfrac{1}{n} + \tfrac{1}{n}$$
$$= \frac{2}{n} \qquad \forall n \geq k$$

حسب خاصية أرخميدس $\forall\, \epsilon > 0$ يوجد $n \in N$ بحيث

$$\frac{2}{n} < \epsilon$$

let $k = n$ when $k \in N$ so

$$\| x_n - x_0 \| = \frac{2}{n} < \epsilon \qquad \forall n > k$$

(2) $y_n \longrightarrow y_0 = (1, 0, 2)$

$$\| y_n - y_0 \| = \| (1, -\tfrac{1}{n}, 2) - (1, 0, 2) \|$$
$$= \| (0, -\tfrac{1}{n}, 0) \|$$
$$= |0| + |-\tfrac{1}{n}| + |0| \qquad \exists\, k \in N$$
$$= \tfrac{1}{n} < \epsilon \qquad \forall n > k$$

(3) $x_n + y_n \longrightarrow x_0 + y_0$

$$x_n + y_n = (0, 1+\tfrac{1}{n}, \tfrac{1}{n}) + (1, -\tfrac{1}{n}, 2)$$
$$= (1, 1, 2+\tfrac{1}{n})$$

$$x_0 + y_0 = (0, 1, 0) + (1, 0, 2)$$
$$= (1, 1, 2)$$

$$\|(x_n + y_n) - (x_0 + y_0)\| = \|(1, 1, 2+\tfrac{1}{n}) - (1, 1, 2)\|$$
$$= \|(0, 0, \tfrac{1}{n})\|$$
$$= |0| + |0| + |\tfrac{1}{n}|$$
$$= \tfrac{1}{n} < \epsilon \qquad \forall n > k$$

(4) $$x_n - y_n = (0, 1+\tfrac{1}{n}, \tfrac{1}{n}) - (1, \tfrac{-1}{n}, 2)$$
$$= (-1, 1+\tfrac{2}{n}, \tfrac{1}{n} - 2)$$
$$x_0 - y_0 = (0, 1, 0) - (1, 0, 2)$$
$$= (-1, 1, -2)$$
$$\|(x_n - y_n) - (x_0 - y_0)\| = \|(-1, 1+\tfrac{2}{n}, \tfrac{1}{n} - 2) - (-1, 1, -2)\|$$
$$= \|0, \tfrac{2}{n}, \tfrac{1}{n}\|$$
$$= |0| + |\tfrac{2}{n}| + |\tfrac{1}{n}|$$
$$= \tfrac{3}{n} < \epsilon \qquad \forall n > k$$

(5) $$3x_n = 3(0, 1+\tfrac{1}{n}, \tfrac{1}{n})$$
$$= (0, 3+\tfrac{3}{n}, \tfrac{3}{n})$$
$$3x_0 = 3(0, 1, 0) = (0, 3, 0)$$
$$\|3x_n - 3x_0\| = \|(0, 3+\tfrac{3}{n}, \tfrac{3}{n}) - (0, 3, 0)\|$$
$$= \|(0, \tfrac{3}{n}, \tfrac{3}{n})\|$$
$$= |0| + |\tfrac{3}{n}| + |\tfrac{3}{n}|$$
$$= \tfrac{6}{n} < \epsilon \qquad \forall n > k$$

(6) $\|x_n\| \longrightarrow \|x_0\|$

$$\big|\, \|x_n\| - \|x_0\| \,\big| = \big|\, \|(0, 1+\tfrac{1}{n}, \tfrac{1}{n})\| - \|(0,1,0)\| \,\big|$$

$$= \big|\, |0| + |1+\tfrac{1}{n}| + |\tfrac{1}{n}| - \{|0| + |1| + |0|\} \,\big|$$

$$= \big|\, 1 + \tfrac{1}{n} + \tfrac{1}{n} - 1 \,\big|$$

$$= \frac{2}{n} < \epsilon \qquad \forall n > k$$

(7) $x_n - y_n = (-1, 1+\frac{2}{n}, \frac{1}{n} - 2)$

$x_0 - y_0 = (-1, 1, -2)$

H.W.

Cauchy Sequence

A sequence $(x_n)_{n=1}^{\infty}$ in a normed space X is called Cauchy, if for every $\epsilon > 0$ there is K such that

$$\|x_n - x_m\| < \epsilon \quad \forall m, n > K.$$

Example The sequence $(\frac{1}{n})_{n=1}^{\infty}$ in $\mathbb{R}$ is a Cauchy sequence.

$$\left|\frac{1}{n} - \frac{1}{m}\right| = \left|\frac{n-m}{nm}\right| = \frac{|n-m|}{nm}$$

by Archandis properties

$\epsilon > 0 \quad b = |n-m|$, then exist $K \in \mathbb{N}$

such tha $\quad \frac{b}{K} < \epsilon$

$$\text{so } \left|\frac{1}{n} - \frac{1}{m}\right| = \frac{|n-m|}{nm} < \epsilon \quad \forall n, m > K$$

$\therefore (\frac{1}{n})_{n=1}^{\infty}$ is a Cauchy sequence.

Lemma every convergent sequence in a normed space $(X, \|\cdot\|)$ is a Cauchy sequence.

proof: Let $(x_n)_{n\geq1}^{\infty}$ be a convergent sequence in a normed space $(X, \|\cdot\|)$ such that $x_n \to x_0$ that is mean for all $\frac{\epsilon}{2} > 0$ there exists $k \in N$ such that

$$\|x_n - x_0\| < \frac{\epsilon}{2} \quad \forall n > k.$$

now

$$\begin{aligned}\|x_n - x_m\| &= \|(x_n - x_0) + (x_0 - x_m)\| \\ &\leq \|x_n - x_0\| + \|x_0 - x_m\| \\ &= \|x_n - x_0\| + \|x_m - x_0\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall m, n > k\end{aligned}$$

$\therefore (x_n)_{n\geq1}^{\infty}$ is a Cauchy sequence.

bounded sequence

Let $(x_n)_{n\geq1}^{\infty}$ be a sequence in a normed space $(X, \|\cdot\|)$ we say that $(x_n)_{n\geq1}^{\infty}$ is a bounded sequence if then exist a positive number c such that

$\|x_n\| < c$ for $n \in N$

Theorem

If $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in a normed space X, then $(x_n)_{n=1}^{\infty}$ is a bounded.

proof,

Let $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence in a normed space $(X, \|\cdot\|)$ for all $\epsilon > 0$, $\exists k \in N$ such that

$\|x_n - x_m\| < \epsilon \quad \forall m, n > k$

take $\epsilon = 1$

$\|x_n - x_m\| < 1 \quad \forall m, n > k$

Let $m = k+1$

$\|x_n - x_{k+1}\| < 1$

$| \|x_n\| - \|x_{k+1}\| | < \|x_n - x_{k+1}\| < 1$

either $\|x_n\| > \|x_{k+1}\|$ or $\|x_n\| < \|x_{k+1}\|$

$\|x_n\| - \|x_{k+1}\| < 1$

Then $\|x_n\| < 1 + \|x_{k+1}\|$ $\qquad \|x_n\| < \|x_{k+1}\|$

we have $\|x_n\| < 1 + \|x_{k+1}\| \quad \forall n > k$

Let $C = 1 + \max\{\|x_1\|, \|x_2\|, \ldots \|x_{k+1}\|\}$

so $\|x_n\| \leq C$

$\forall n \in N$

Example Let $X = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \ a, b, c \in \mathbb{R} \right\}$

$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$

$\|A\|_\infty = \max\{|a|, |b|, |c|\}$

$x_n = \begin{bmatrix} 2 & \frac{1}{n}+1 \\ 0 & 1-\frac{1}{n} \end{bmatrix}$

is $(x_n)_{n=1}^{\infty}$ is bounded

$\|x_n\| = \max\{|2|, |\frac{1}{n}+1|, |1-\frac{1}{n}|\}$

≤ 2

So $\|x_n\| \leq 2 \quad \forall n \in \mathbb{N}$

$\therefore (x_n)_{n=1}^{\infty}$ is bounded

$y_n = \begin{bmatrix} n+1 & \frac{1}{n} \\ 0 & 3 \end{bmatrix}$

$\|y_n\| = \max\{|n+1|, |\frac{1}{n}|, |3|\}$

not exist $c \in \mathbb{R}^+$ such that

$n+1 < c \quad \forall n \in \mathbb{N}$

So $(y_n)_{n=1}^{\infty}$ is not bounded.