now take To show that T is ~~unit~~ normal

$$TT^*(x) = T(T^*(x))$$
$$= T(-2ix)$$
$$= -2i\,T(x)$$
$$= -2i\;2ix$$
$$= -4i^2x$$
$$= 4x$$

$\therefore TT^*(x) = 4x \quad \forall x \in H$

now $T^*T(x) = T^*(T(x))$
$$= T^*(2ix)$$
$$= 2i\,T^*(x)$$
$$= 2i(-2ix)$$
$$= -4i^2x$$
$$= 4x$$

$\therefore T^*T(x) = 4x$

So $TT^*(x) = T^*T(x) \quad \forall x \in H$

$\therefore TT^* = T^*T$

So T is normal operator.

Example Let $T: \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ ($\mathbb{C}^2$ is over $\mathbb{C}$) be an operator

defined by $T(x) = T(x_1, x_2) = (x_2, x_1)$

find T^* if exist and which any type of self-adjoint, unitary, normal are belong?

Solution At first must show that T is bounded linear operator.

Let $\alpha, \beta \in \mathbb{C}$, $x, y \in \mathbb{C}^2$ such that $x = (x_1, x_2)$, $y = (y_1, y_2)$, $x_1, x_2, y_1, y_2 \in \mathbb{C}$

$$\begin{aligned}
T(\alpha x + \beta y) &= T(\alpha(x_1, x_2) + \beta(y_1, y_2)) \\
&= T((\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2)) \\
&= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2) \\
&= (\alpha x_2 + \beta y_2, \alpha x_1 + \beta y_1) \\
&= (\alpha x_2, \alpha x_1) + (\beta y_2, \beta y_1) \\
&= \alpha(x_2, x_1) + \beta(y_2, y_1) \\
&= \alpha T(x_1, x_2) + \beta T(y_1, y_2) \\
&= \alpha T(x) + \beta T(y)
\end{aligned}$$

$\therefore$ T is linear operator

~~$\|T(x)\|$~~ we to show that T is bounded

Let $x = (x_1, x_2) \in \mathbb{C}^2$ where $x_1, x_2 \in \mathbb{C}$

$$\begin{aligned}
\|T(x)\|^2 &= \langle T(x) / T(x) \rangle \\
&= \langle T(x_1, x_2) / T(x_1, x_2) \rangle \\
&= \langle (x_2, x_1) / (x_2, x_1) \rangle \\
&= x_2 \overline{x_2} + x_1 \overline{x_1} \\
&= x_1 \overline{x_1} + x_2 \overline{x_2} = \langle (x_1, x_2) / (x_1, x_2) \rangle \\
&= \|x\|^2
\end{aligned}$$

(6)

$\therefore \|T(x)\|^2 = \|x\|^2$

$\|T(x)\| = \|x\| \quad \forall x \in \mathbb{C}^2$

So T is bounded with $c = 1$

$$\|T\| = \sup_{0 \neq x \in \mathbb{C}^2} \left\{ \frac{\|T(x)\|}{\|x\|} \right\} = \sup_{0 \neq x \in \mathbb{C}^2} \left\{ \frac{\|x\|}{\|x\|} \right\} = 1$$

$\therefore \|T\| = 1$

so T^* is exist and unique bounded linear and $\|T\| = \|T^*\|$

$$\langle T(x) / y \rangle = \langle x / T^*(y) \rangle$$

~~$\langle T(x_1, x_2) / (y_1, y_2) \rangle$~~

$$\begin{aligned} \langle T(x) / y \rangle &= \langle T(x_1, x_2) / (y_1, y_2) \rangle \\ &= \langle (x_2, x_1) / (y_1, y_2) \rangle \\ &= x_2 \overline{y_1} + x_1 \overline{y_2} \\ &= x_1 \overline{y_2} + x_2 \overline{y_1} \\ &= \langle (x_1, x_2) / (y_2, y_1) \rangle \\ &= \langle x / (y_2, y_1) \rangle \end{aligned}$$

$$\therefore \langle x / (y_2, y_1) \rangle = \langle x / T^*(y) \rangle$$

we get that $T^*(y) = (y_2, y_1)$

that is mean $T^*(x) = T^*(x_1, x_2) = (x_2, x_1)$

$$\therefore T(x) = T(x_1, x_2) = (x_2, x_1) = T^*(x_1, x_2) = T^*(x)$$

$\therefore T(x) = T^*(x) \quad \forall x \in X$

$\therefore T = T^*$ $\quad$ T is self adjoint operator now to show that if T is unitary

$T(x) = T(y)$

$T(x_1, x_2) = T(y_1, y_2)$

$(x_2, x_1) = (y_2, y_1)$

$\Downarrow \; x_2 = y_2$
$\quad x_1 = y_1$

So $(x_1, x_2) = (y_1, y_2)$

$\therefore \; x = y$

So T is one to one

$\forall\, y \in$ Codomain

~~$\exists\, x \in$ domain~~

$y = (y_1, y_2)$

$\exists\, w \in H$, domain

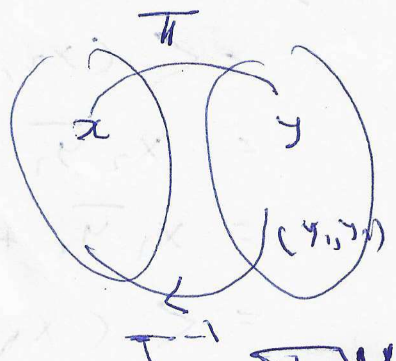
$w = (y_2, y_1)$ such that

$T(w) = T(y_2, y_1) = (y_1, y_2)$
$= y$

So T is onto.

$\therefore$ T is bijective operator.

So T^{-1} is exist



$w = (x_2, x_1)$

$T(w) = x$

$T(x_2, x_1) = (x_1, x_2)$

$\cancel{T^{-1}}\, T(x_2, x_1) = T^{-1}(x_1, x_2)$

$(x_2, x_1) = T^{-1}(x)$

~~$T(y) = ($~~

~~$T(x_2, x_1) = (x_1, x_2)$~~

~~$T^{-1}T(x_2, x_1) = T^{-1}(x_1, x_2)$~~

So $T^{-1}(x) = (x_2, x_1) = T(x_1, x_2) = T(x) = T^{*}(x)$

$\therefore \; T^{*}(x) = T^{-1}(x) \quad \forall\, x \in \mathbb{C}^2$

$\therefore$ T is unitary operator.

$\therefore$ T is normal operator since it is self adjoint

$$T T^*(x) = T T^*(x_1, x_2)$$
$$= T(x_2, x_1)$$
$$= (x_1, x_2)$$
$$T T^*(x) = x$$

$$T^* T(x) = T^*(T(x))$$
$$= T^*(T(x_1, x_2))$$
$$= T^*(x_2, x_1)$$
$$= (x_1, x_2)$$
$$= x$$

$\therefore$ $T T^*(x) = T^* T(x) \quad \forall x \in \mathbb{C}^2$

$\therefore$ $T T^* = T^* T$

$\therefore$ T is normal operator.

للفصل ح 2.2

Example Let $T: \mathbb{C}^2 \to \mathbb{C}$ be an operator defined by $T(x) = T(x_1, x_2) = x_1$.

Find T^* if exists and which any type of self adjoint, unitary, normal are belong?

Solution At first must show that T is bounded linear operator.

Let $\alpha, \beta \in \mathbb{C}$, $x, y \in \mathbb{C}^2$ such that

$x = (x_1, x_2)$ $\quad x_1, x_2, y_1, y_2 \in \mathbb{C}$

$y = (y_1, y_2)$

$$T(\alpha x + \beta y) = T(\alpha(x_1, x_2) + \beta(y_1, y_2))$$
$$= T((\alpha x_1, \alpha x_2) + (\beta y_1, \beta y_2))$$
$$= T(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)$$
$$= \alpha x_1 + \beta y_1$$
$$= \alpha T(x_1, x_2) + \beta T(y_1, y_2)$$
$$= \alpha T(x) + \beta T(y)$$

$\therefore T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

So T is linear operator.

we to show that T is bounded. Let $x \in \mathbb{C}^2$

$x = (x_1, x_2)$, $x_1, x_2 \in \mathbb{C}$

$$\|T(x)\|^2 = \langle T(x) / T(x) \rangle$$
$$= \langle T(x_1, x_2) / T(x_1, x_2) \rangle$$
$$= \langle x_1 / x_1 \rangle$$
$$= x_1 \overline{x_1} \quad \text{it is real number positive}$$
$$\leq x_1 \overline{x_1} + x_2 \overline{x_2} \quad \text{It is real positive number}$$
$$= \langle (x_1, x_2) / (x_1, x_2) \rangle$$
$$= \|(x_1, x_2)\|^2$$
$$= \|x\|^2$$

$\therefore \|T(x)\|^2 \leq \|x\|^2 \quad \forall x \in \mathbb{C}^2$

So $\|T(x)\| \leq \|x\|$

T is bounded

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For find $\|T\|$

$$\|T\| = \sup_{0 \neq x \in \mathbb{C}^2} \left\{ \frac{\|T(x)\|}{\|x\|} \right\} \leq \sup_{0 \neq x \in \mathbb{C}^2} \left\{ \frac{\|x\|}{\|x\|} \right\} = 1$$

now since T is bounded

$$\|T(x)\| \leq \|T\| \, \|x\| \quad \forall x \in \mathbb{C}^2.$$

Let $x = (x_1, 0)$ in special cause

$$\cancel{\|T(x)\|} \quad \cancel{\|T(x)\|^2 \leq \|T\|^2 \|x\|^2}$$

$$\cancel{\frac{\|T(x)\|^2}{\|x\|^2} \leq \|T\|}$$

$$\sqrt{\|x\|^2} = \sqrt{\langle x / x \rangle}$$

$$= \sqrt{\langle (x_1, 0) / (x_1, 0) \rangle}$$

$$= \sqrt{x_1 \bar{x}_1 + \cdots}$$

$$= \sqrt{|x_1|^2}$$

$$= |x_1|$$

$$\frac{\|T(x)\|}{\|x\|} \leq \|T\|$$

$$\frac{\|T(x_1, 0)\|}{\sqrt{\|x\|^2}} \leq \|T\|$$

$$\frac{\|x_1\|}{\sqrt{\langle (x_1,0) / (x_1,0) \rangle}} \leq \|T\|$$

$$\frac{|x_1|}{|x_1|} \leq \|T\|$$

$$1 \leq \|T\|$$

So $\|T\| = 1$

so T^* is exist unique bounded linear and $\|T\| \leq \|T^*\|$

Let $x \in \mathbb{C}^2$, $y \in \mathbb{C}$

$$\langle T(x) / y \rangle = \langle x / T^*(y) \rangle$$

$$\langle T(x)/y \rangle = \langle T(x_1, x_2)/y \rangle$$
$$= \langle x_1 / y \rangle$$
$$= x_1 \bar{y} \quad \text{--- (1)}$$

Let $T^* : \mathbb{C} \longrightarrow \mathbb{C}^2$

$$T^*(y) = w = (w_1, w_2)$$

$$\langle x / T^*(y) \rangle = \langle x / w \rangle$$
$$= \langle (x_1, x_2) / (w_1, w_2) \rangle$$
$$= x_1 \bar{w_1} + x_2 \bar{w_2} \quad \text{--- (2)}$$

Since (1) = (2) we get

$$x_1 \bar{y} = x_1 \bar{w_1} + x_2 \bar{w_2}$$

we get $w_2 = 0$ the equality with

$w_1 = y$

so $T^*(y) = (y, 0)$