Note:- $\forall x, y \in \mathbb{R}$, we have $2xy \leq x^2 + y^2$.

Proof (2):- let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, $a_1, a_2, b_1, b_2 \in \mathbb{R}$

$$|z_1 + z_2| = |(a_1 + ib_1) + (a_2 + ib_2)| = |(a_1 + a_2) + i(b_1 + b_2)|$$

$$\therefore |z_1 + z_2|^2 = |(a_1 + a_2) + i(b_1 + b_2)|^2$$

$$= \left(\sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}\right)^2$$

$$= (a_1 + a_2)^2 + (b_1 + b_2)^2$$

$$= a_1^2 + 2a_1a_2 + a_2^2 + b_1^2 + 2b_1b_2 + b_2^2$$

$$\therefore |z_1 + z_2|^2 = a_1^2 + b_1^2 + a_2^2 + b_2^2 + 2(a_1a_2 + b_1b_2) \text{ ---- (a)}$$

we see that $(a_1a_2 + b_1b_2)^2 = a_1^2a_2^2 + 2a_1a_2b_1b_2 + b_1^2b_2^2$

let $x = a_1b_2$ $\quad y = a_2b_1$

$$\because 2xy \leq x^2 + y^2$$

$$\therefore 2a_1b_2a_2b_1 \leq a_1^2b_2^2 + a_2^2b_1^2$$

$$\Rightarrow (a_1a_2 + b_1b_2)^2 \leq a_1^2a_2^2 + (a_1^2b_2^2 + a_2^2b_1^2) + b_1^2b_2^2$$

$$\Rightarrow (a_1a_2 + b_1b_2)^2 \leq (a_1^2 + b_1^2)(a_2^2 + b_2^2)$$

الجذر التربيعي للطرفين

$$\Rightarrow (a_1a_2 + b_1b_2) \leq \sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)} \text{ ---- (b)}$$

$\Rightarrow$

نعوض (b) في (a)

$$\because |z_1+z_2|^2 = a_1^2+b_1^2+a_2^2+b_2^2+2(a_1a_2+b_1b_2)$$

$$|z_1+z_2|^2 \le a_1^2+b_1^2+a_2^2+b_2^2+2\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}$$

$$\le a_1^2+b_1^2+a_2^2+b_2^2+2\sqrt{(a_1^2+b_1^2)}\sqrt{(a_2^2+b_2^2)}$$

$$\le |z_1|^2+|z_2|^2+2|z_1||z_2|$$

$$\le (|z_1|+|z_2|)^2$$ جذر الطرفين

$$\therefore |z_1+z_2| \le |z_1|+|z_2|.$$

Proof(3):- $|z_1| = |z_2+(z_1-z_2)|$

$$|z_1| \le |z_2| + |z_1-z_2|$$ من (a)

$$|z_1|-|z_2| \le |z_1-z_2|$$

by the same way

$$|z_2| = |z_1+(z_2-z_1)|$$

$$\le |z_1|+|z_2-z_1|$$

$$|z_2|-|z_1| \le |z_2-z_1|.$$

Note:- for all finite number $z_1, z_2, \ldots, z_n$ of the complex number then

$$|z_1 z_2 \cdots z_n| = |z_1||z_2||z_3| \cdots |z_n|$$

2. If $z_1, z_2 \in \mathbb{C}$, $z_2 \neq 0$, then $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

Proof:- put $w = \frac{z_1}{z_2}$, that is $z_1 = w z_2$

$$\Rightarrow |z_1| = |w z_2| = |w||z_2|$$

$$\Rightarrow |w| = \frac{|z_1|}{|z_2|} \Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

Def.:- let $z \in \mathbb{C}$ and $z = a + ib$, $a, b \in \mathbb{R}$. Then the conjugat number of z is $\overline{z} = a - ib$.

EX.:- $z = 2 + 3i \longrightarrow \overline{z} = 2 - 3i$

$z = 3 \longrightarrow \overline{z} = 3$

$z = -5i \longrightarrow \overline{z} = 5i$

The properties of conjugat number

1. $z + \overline{z} = 2\,Re(z) = 2a$.
2. $z - \overline{z} = 2\,Im(z) = 2bi$.
3. $z\overline{z} = a^2 + b^2 = |z|^2$.
4. $\overline{\overline{z}} = z$.
5. $(\overline{z_1 + z_2}) = \overline{z_1} + \overline{z_2}$.
6. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$.
7. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.
8. $|z| = |\overline{z}|$.

Lemma:- If $z \neq 0$, then $z^{-1} = \frac{\bar{z}}{|z|^2}$

proof:- let $z = a + ib \neq 0$, then $\bar{z} = a - ib$

$$z^{-1} = \frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib}$$

$$z^{-1} = \frac{a-ib}{a^2+b^2} = \frac{\bar{z}}{|z|^2}.$$ من الخاصية (e)

EX:- prove that $\left|\sum_{n=1}^{k} z_n\right| \leq \sum_{n=1}^{k} |z_n|$

proof:- let $k=1 \Rightarrow |z_1| = |z_1|$

$k=2 \Rightarrow |z_1+z_2| \leq |z_1| + |z_2|$

suppose that for $k=m \Rightarrow \left|\sum_{k=1}^{m} z_k\right| \leq |z_1| + |z_2| + \cdots + |z_m|$

$$\leq \sum_{k=1}^{m} |z_k|$$

for $k = m+1$

$$\left|\sum_{k=1}^{m+1} z_k\right| = |z_1 + z_2 + \cdots + z_{m+1}|$$

$$\leq |z_1| + |z_2| + \cdots + |z_m| + |z_{m+1}|$$

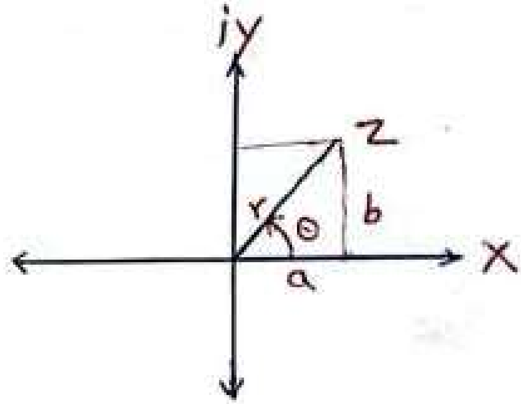
$$\leq \sum_{k=1}^{m+1} |z_k|$$

$\therefore$ by Mathmatical induction

$$\left|\sum_{n=1}^{k} z_n\right| \leq \sum_{n=1}^{k} |z_n|.$$

The polar representation for the complex numbers:-

نفرض انه لدينا العدد العقدي $Z=a+ib$ فعند تمثيل Z بنقطة في المستوي xy فإن مقياس العدد العقدي هو المسافة او الطول r من نقطة الاصل الى النقطة Z ، نفرض ان العدد العقدي Z يصنع زاوية θ ونرمز لها بالرمز $\alpha=\arg(Z)$ بين الاتجاه الموجب للمحور الحقيقي وبين المتجه الذي يمثل Z .



$\therefore r=\sqrt{a^2+b^2}$

$\tan\theta=\frac{b}{a} \Rightarrow \tan^{-1}\frac{b}{a}=\theta$

$\therefore Z=r(\cos\theta+i\sin\theta)$

$=r(\cos\theta, \sin\theta)$

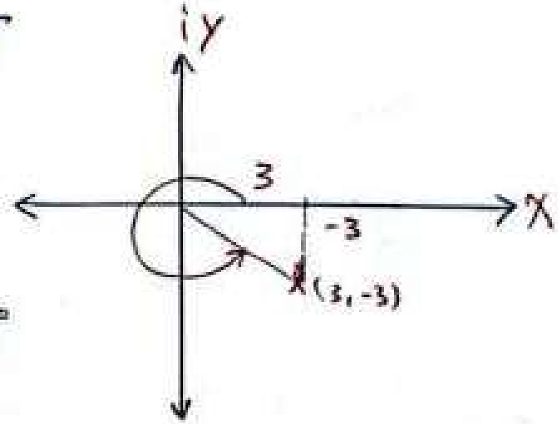
$=re^{i\theta}$ اويلر

$a=r\cos\theta \quad , \quad b=r\sin\theta \Rightarrow \cos\theta=\frac{a}{r} \quad , \quad \sin\theta=\frac{b}{r}$

Ex.:- Find the polar representation of :

1. $Z=3-3i$ 2. $Z=-4$ 3. $Z=\frac{-1}{2}+\frac{\sqrt{3}}{2}i$ 4. $Z=2\sqrt{3}-2i$.

Sol.(1):- $r=\sqrt{a^2+b^2}=\sqrt{3^2+3^2}=3\sqrt{2}$



$\tan\theta=\frac{b}{a}=\frac{-3}{3}=-1$

$\theta=\tan^{-1}(-1)$

$\arg(Z)=2\pi-\frac{\pi}{4}=\frac{7\pi}{4}=315^\circ$

$\therefore Z=3\sqrt{2}(\cos 315^\circ+i\sin 315^\circ)$.