Corollary. If $(H', \circ)$ is any normal subgroup of the group $(G', \circ)$, then $(G/f^{-1}(H'), \otimes) \simeq (G'/H', \otimes')$.

Proof. By the corollary to Theorem 2–39, the pair $(f^{-1}(H'), *)$ is a normal subgroup of $(G, *)$. Moreover, $\ker (f) \subseteq f^{-1}(H')$, so the hypothesis of the theorem is completely satisfied. This leads to the isomorphism

$$(G/f^{-1}(H'), \otimes) \simeq \big(G'/f(f^{-1}(H')), \otimes'\big).$$

Since f is a mapping onto G', $H' = f(f^{-1}(H'))$, and we are done.

Our final theorem, a rather technical result, will be crucial to the proof of the Jordan-Hölder Theorem.

Theorem 2–51. If $(H, *)$ and $(K, *)$ are subgroups of the group $(G, *)$ with $(K, *)$ normal, then $(H/H \cap K, \otimes) \simeq (H * K/K, \otimes')$.

Proof. Needless to say, it should be checked that the quotient groups appearing in the statement of the theorem are actually defined. We leave to the reader the routine task of verifying that $(H \cap K, *)$ is a normal subgroup of $(H, *)$, that $(H * K, *)$ is a group, and that $(K, *)$ is normal in $(H * K, *)$.

Our proof is patterned on that of Theorem 2–50. Here, the problem is to construct a homomorphism α from the group $(H, *)$ onto the quotient group $(H * K/K, \otimes')$ for which $\ker (\alpha) = H \cap K$. To achieve this, consider the function $\alpha(h) = h * K$, $h \in H$. Note, $H = H * e \subseteq H * K$, so that α can be obtained by composing the inclusion map $i_H: H \to H * K$ with the natural mapping $\mathrm{nat}_K: H * K \to H * K/K$. In other words,

$$\alpha = \mathrm{nat}_K \circ i_H,$$

or, in diagrammatic language,

$$\begin{tikzcd} H \arrow[rr, "i_H"] \arrow[dr, "\alpha"'] & & H * K \arrow[dl, "\mathrm{nat}_K"] \\ & H * K/K & \end{tikzcd}$$

The foregoing factorization implies α is a homomorphism and $\alpha(H) = H * K/K$.

We next proceed to establish that the kernel of α is precisely the set $H \cap K$. First, observe that the coset $K = e * K$ serves as the identity element of the quotient group $(H * K/K, \otimes')$. This means

$$\begin{aligned} \ker (\alpha) &= \{h \in H \mid \alpha(h) = K\} = \{h \in H \mid h * K = K\} \\ &= \{h \in H \mid h \in K\} = H \cap K. \end{aligned}$$

The required isomorphism is now evident from the Fundamental Theorem.

Example 2–54. As an illustration of this last result, let us return again to the group of integers $(Z, +)$ and consider the cyclic subgroups $((3), +)$ and $((4), +)$. Both these subgroups are normal, since $(Z, +)$ is a commutative group. Moreover, it is fairly obvious that

$$(3) \cap (4) = (12) \qquad \text{and} \qquad (3) + (4) = Z.$$

Theorem 2–51 then tells us that

$$((3)/(12), \otimes) \simeq (Z/(4), \otimes'),$$

where $\otimes$ and $\otimes'$ designate the respective quotient group operations.

The notation tends to obscure the simplicity of our conclusion. For the reader will doubtless recall that $(Z/(4), \otimes')$ is just the group of integers modulo 4. A closer examination of the cosets and operation in the system $((3)/(12), \otimes)$ reveals this to be nothing more than the group $(\{0, 3, 6, 9\}, +_{12})$. What we actually have is a disguised version of the isomorphism,

$$(\{0, 3, 6, 9\}, +_{12}) \simeq (Z_4, +_4).$$

PROBLEMS

1. Let $(G, *)$ be the group of symmetries of the square and $(G', \circ)$ be the Klein four-group (see Examples 2–24 and 2–47). The following mapping defines a homomorphism from $(G, *)$ onto $(G', \circ)$:

$$f(R_{180}) = f(R_{360}) = e, \qquad f(R_{90}) = f(R_{270}) = a,$$
$$f(H) = f(V) = b, \qquad f(D_1) = f(D_2) = c.$$

   a) Establish the isomorphism $(G/\text{cent } G, \otimes) \simeq (G', \circ)$.
   b) Write out the induced mapping $\bar{f}\colon G/\text{cent } G \to G'$ which leads to this isomorphism.

2. Prove the following generalization of the Factor Theorem: Let f_1 and f_2 be homomorphisms from the group $(G, *)$ onto the groups $(G_1, *_1)$ and $(G_2, *_2)$, respectively. If $\ker(f_1) \subseteq \ker(f_2)$, then there exists a unique homomorphism $\bar{f}\colon G_1 \to G_2$ satisfying $f_2 = \bar{f} \circ f_1$. Diagrammatically,

$$\begin{tikzcd} G \arrow[rr, "f_2"] \arrow[dr, "f_1"'] & & G_2 \\ & G_1 \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

[*Hint:* Use nearly the same argument as for Theorem 2–46; that is, for any element $f_1(a) \in G_1$, define $\bar{f}$ by $\bar{f}(f_1(a)) = f_2(a)$.]

3. Deduce the Factor Theorem from the result of Problem 2.
4. Given a group $(G, *)$, establish the following facts regarding the inner automorphisms of G.
   a) Whenever $(G, *)$ is noncommutative, $I(G) \neq \{i_G\}$.
   b) If $(H, *)$ is a subgroup of $(G, *)$, then the pair $(\sigma_a(H), *)$ is also a subgroup for every $a \in G$.
   c) A subgroup $(H, *)$ is normal in $(G, *)$ if and only if the set H is mapped into itself by each inner automorphism of G.
   d) If the element $x \in G$ is such that $\sigma_a(x) = x$ for every $a \in G$, then the cyclic subgroup $((x), *)$ is normal in $(G, *)$.
5. Let $(G, *)$ be a group and the elements $x, y \in G$. We say x is *conjugate* to y, written $x \sim y$, if and only if $\sigma_a(x) = y$ for some $a \in G$. Prove that conjugation is an equivalence relation in G.
6. Obtain the isomorphism $(S_3, \circ) \simeq (I(S_3), \circ)$.
7. Assume f is a homomorphism from the group $(G, *)$ into itself having the property of commuting with every inner automorphism of G; that is, $\sigma_a \circ f = f \circ \sigma_a$ for every $a \in G$. If the set H is defined by

$$H = \{x \in G \mid f(x) = x\},$$

   prove that
   a) the pair $(H, *)$ is a normal subgroup of $(G, *)$,
   b) the quotient group $(G/H, \otimes)$ is commutative. [*Hint:* $[G, G] \subseteq H$.]
8. Let $(H, *)$ be a normal subgroup of the group $(G, *)$. Further, let $(K_1, *)$ and $(K_2, *)$ be subgroups of $(G, *)$ such that $H \subseteq K_1$, $H \subseteq K_2$. Prove that $K_1 \subseteq K_2$ if and only if $\text{nat}_H\, K_1 \subseteq \text{nat}_H\, K_2$.
9. Given $(G, *)$ is the group of symmetries of the square. Exhibit the correspondence between those subgroups $(H, *)$ of $(G, *)$ with $\text{cent}\, G \subseteq H$ and the subgroups of $(G/\text{cent}\, G, \otimes)$.

In Problems 10 through 14, f denotes a homomorphism from the group $(G, *)$ onto the group $(G', \circ)$.

10. Show that the Factor Theorem implies the function f can be expressed (nontrivially) as the composition of an onto function and a one-to-one function.
11. If $(H, *)$ is a subgroup of $(G, *)$ for which $H = f^{-1}(f(H))$, verify that $\ker(f) \subseteq H$. [*Hint:* $f^{-1}(f(H)) = H * \ker(f)$.]
12. For a proof of Theorem 2-50 that does not depend on the Fundamental Theorem, define the function $g\colon G/H \to G'/f(H)$ by taking

$$g(a * H) = f(a) \circ f(H).$$

   a) Show that g is well-defined, one-to-one, operation-preserving, and onto $G'/f(H)$; hence, $(G/H, \otimes) \simeq (G'/f(H), \otimes')$.
   b) Establish that g is the unique mapping which makes the diagram at the right commutative.

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "\text{nat}_H"'] & G' = f(G) \arrow[d, "\text{nat}_{f(H)}"] \\
G/H \arrow[r, "g"] & G'/f(H)
\end{tikzcd}$$

13. If ker $(f) \subseteq [G, G]$, prove that $(G/[G, G], \otimes) \simeq (G'/[G', G'], \otimes')$.

14. Suppose $(H, *)$ and $(K, *)$ are normal subgroups of the group $(G, *)$ with $H \subseteq K$. Prove that
    a) $(H, *)$ is a normal subgroup of $(K, *)$,
    b) $(K/H, \otimes)$ is a normal subgroup of the quotient group $(G/H, \otimes)$,
    c) $(G/H \,/\, K/H, \otimes') \simeq (G/K, \otimes)$.
    [*Hint:* Apply Theorem 2-50 with nat_H and $(G/H, \otimes)$ replacing f and $(G', \circ)$.]

15. Given $(H, *)$ and $(K, *)$ are subgroups of the group $(G, *)$ with $(K, *)$ normal. Assuming $H \cap K = \{e\}$, derive the isomorphism $(H, *) \simeq (H * K/K, \otimes)$.

16. In the symmetric group $(S_4, \circ)$, let the set K consist of the four permutations

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}, \quad \begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 4 & 3\end{pmatrix}, \quad \begin{pmatrix}1 & 2 & 3 & 4\\ 3 & 4 & 1 & 2\end{pmatrix}, \quad \begin{pmatrix}1 & 2 & 3 & 4\\ 4 & 3 & 2 & 1\end{pmatrix}.$$

    a) Show that the pair $(K, \circ)$ is a normal subgroup of $(S_4, \circ)$.
    b) Establish that $(S_4/K, \otimes)$ and $(S_3, \circ)$ are isomorphic groups. [*Hint:* Define $H = \{f \in S_4 \mid f(4) = 4\}$ and use Problem 15.]

17. Let $(G, *)$ and $(G', \circ)$ be two groups with identity elements e and e', respectively. Recall that their direct product is the group $(G \times G', \cdot)$, with the operation $\cdot$ defined on the Cartesian product $G \times G'$ in the natural way:

$$(a, a') \cdot (b, b') = (a * b, a' \circ b')$$

    for all $(a, a'), (b, b') \in G \times G'$. Prove the following statements:
    a) If $H = G \times e'$ and $H' = e \times G'$, then $(H, \cdot)$ and $(H', \cdot)$ are both normal subgroups of $(G \times G', \cdot)$.
    b) $(H, \cdot) \sim (G, *)$ and $(H', \cdot) \sim (G', \circ)$.
    c) Every element of H commutes with every element of H'.
    d) Each member of $G \times G'$ can be uniquely expressed as the product of an element of H by an element of H'.
    e) $(G \times G'/H, \otimes) \simeq (H', \cdot)$ and $(G \times G'/H', \otimes) \simeq (H, \cdot)$. Derive these isomorphisms in two ways: use the Fundamental Theorem, and then Theorem 2-51.

18. Illustrate the various parts of Problem 17 by considering the direct product of the groups $(Z_3, +_3)$ and $(Z_4, +_4)$. In addition, show that

$$(Z_3 \times Z_4, \cdot) \simeq (Z_{12}, +_{12}).$$

    [*Hint:* $(Z_3 \times Z_4, \cdot)$ is a cyclic group of order 12.]

19. Suppose $(H, *)$ and $(K, *)$ are subgroups of the group $(G, *)$ such that
    1) every element of H commutes with every element of K,
    2) every member of G can be uniquely expressed as the product of an element of H by an element of K.

    Prove that $(G, *) \sim (H \times K, \cdot)$.