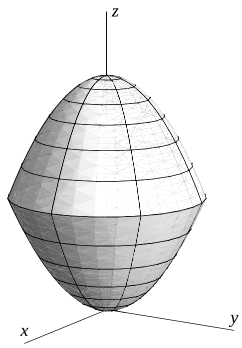
Triple Integrals in Cylindrical or Spherical Coordinates

1. *Let $\mathcal{U}$ be the solid enclosed by the paraboloids $z = x^2 + y^2$ and $z = 8 - (x^2 + y^2)$. (Note: The paraboloids intersect where $z = 4$.) Write $\iiint_{\mathcal{U}} xyz\ dV$ as an iterated integral in cylindrical coordinates.*



Solution. This is the same problem as #3 on the worksheet "Triple Integrals", except that we are now given a specific integrand. It makes sense to do the problem in cylindrical coordinates since the solid is symmetric about the z-axis. In cylindrical coordinates, the two paraboloids have equations $z = r^2$ and $z = 8 - r^2$. In addition, the integrand xyz is equal to $(r\cos\theta)(r\sin\theta)z$.

Let's write the inner integral first. If we imagine sticking vertical lines through the solid, we can see that, along any vertical line, z goes from the bottom paraboloid $z = r^2$ to the top paraboloid $z = 8 - r^2$. So, our inner integral will be $\int_{r^2}^{8-r^2} (r\cos\theta)(r\sin\theta)z\ dz$.

To write the outer two integrals, we want to describe the projection of the solid onto the xy-plane. As we had figured out last time, the projection was the disk $x^2 + y^2 \leq 4$. We can write an iterated integral in polar coordinates to describe this disk: the disk is $0 \leq r \leq 2$, $0 \leq \theta < 2\pi$, so our iterated integral will just be $\int_0^{2\pi} \int_0^2 (\text{inner integral}) \cdot r\ dr\ d\theta$. Therefore, our final answer is $\boxed{\int_0^{2\pi} \int_0^2 \int_{r^2}^{8-r^2} (r\cos\theta)(r\sin\theta)z \cdot r\ dz\ dr\ d\theta}$.

2. *Find the volume of the solid ball $x^2 + y^2 + z^2 \leq 1$.*

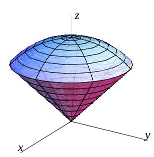
Solution. Let $\mathcal{U}$ be the ball. We know by #1(a) of the worksheet "Triple Integrals" that the volume of $\mathcal{U}$ is given by the triple integral $\iiint_{\mathcal{U}} 1\ dV$. To compute this, we need to convert the triple integral to an iterated integral.

The given ball can be described easily in spherical coordinates by the inequalities $0 \leq \rho \leq 1$, $0 \leq \phi \leq \pi$, $0 \leq \theta < 2\pi$, so we can rewrite the triple integral $\iiint_{\mathcal{U}} 1\ dV$ as an iterated integral in spherical coordinates

$$\boxed{\int_0^{2\pi}\int_0^{\pi}\int_0^1 1\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta} = \int_0^{2\pi}\int_0^{\pi}\left(\frac{\rho^3}{3}\sin\phi\Big|_{\rho=0}^{\rho=1}\right)d\phi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi}\frac{1}{3}\sin\phi\,d\phi\,d\theta$$
$$= \int_0^{2\pi}\left(-\frac{1}{3}\cos\phi\Big|_{\phi=0}^{\phi=\pi}\right)d\theta$$
$$= \int_0^{2\pi}\frac{2}{3}\,d\theta$$
$$= \boxed{\frac{4}{3}\pi}$$

3. *Let $\mathcal{U}$ be the solid inside both the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=1$. Write the triple integral $\iiint_{\mathcal{U}} z\,dV$ as an iterated integral in spherical coordinates.*

   Solution. Here is a picture of the solid:

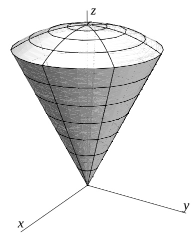
   We have to write both the integrand (z) and the solid of integration in spherical coordinates. We know that z in Cartesian coordinates is the same as $\rho\cos\phi$ in spherical coordinates, so the function we're integrating is $\rho\cos\phi$.

   The cone $z=\sqrt{x^2+y^2}$ is the same as $\phi=\frac{\pi}{4}$ in spherical coordinates.(1) The sphere $x^2+y^2+z^2=1$ is $\rho=1$ in spherical coordinates. So, the solid can be described in spherical coordinates as $0\le\rho\le 1$, $0\le\phi\le\frac{\pi}{4}$, $0\le\theta\le 2\pi$. This means that the iterated integral is $\boxed{\int_0^{2\pi}\int_0^{\pi/4}\int_0^1(\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta}$.

For the remaining problems, use the coordinate system (Cartesian, cylindrical, or spherical) that seems easiest.

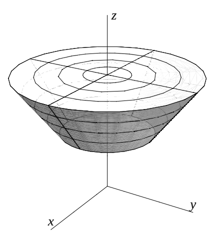
4. *Let $\mathcal{U}$ be the "ice cream cone" bounded below by $z=\sqrt{3(x^2+y^2)}$ and above by $x^2+y^2+z^2=4$. Write an iterated integral which gives the volume of $\mathcal{U}$. (You need not evaluate.)*

(1)Why? We could first rewrite $z=\sqrt{x^2+y^2}$ in cylindrical coordinates: it's $z=r$. In terms of spherical coordinates, this says that $\rho\cos\phi=\rho\sin\phi$, so $\cos\phi=\sin\phi$. That's the same as saying that $\tan\phi=1$, or $\phi=\frac{\pi}{4}$.



Solution. We know by #1(a) of the worksheet "Triple Integrals" that the volume of $\mathcal{U}$ is given by the triple integral $\iiint_{\mathcal{U}} 1\ dV$. The solid $\mathcal{U}$ has a simple description in spherical coordinates, so we will use spherical coordinates to rewrite the triple integral as an iterated integral. The sphere $x^2+y^2+z^2=4$ is the same as $\rho=2$. The cone $z=\sqrt{3(x^2+y^2)}$ can be written as $\phi=\frac{\pi}{6}$.(2) So, the volume is $\boxed{\int_0^{2\pi}\int_0^{\pi/6}\int_0^2 1\cdot\rho^2\sin\phi\ d\rho\ d\phi\ d\theta}$.

5. *Write an iterated integral which gives the volume of the solid enclosed by* $z^2=x^2+y^2$, $z=1$, *and* $z=2$. *(You need not evaluate.)*

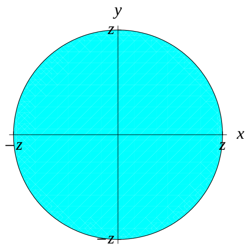


Solution. We know by #1(a) of the worksheet "Triple Integrals" that the volume of $\mathcal{U}$ is given by the triple integral $\iiint_{\mathcal{U}} 1\ dV$. To compute this, we need to convert the triple integral to an iterated integral. Since the solid is symmetric about the z-axis but doesn't seem to have a simple description in terms of spherical coordinates, we'll use cylindrical coordinates.

Let's think of slicing the solid, using slices parallel to the xy-plane. This means we'll write the outer integral first. We're slicing $[1,2]$ on the z-axis, so our outer integral will be $\int_1^2$ something dz.

To write the inner double integral, we want to describe each slice (and, within a slice, we can think of z as being a constant). Each slice is just the disk enclosed by the circle $x^2+y^2=z^2$, which is a circle of radius z:

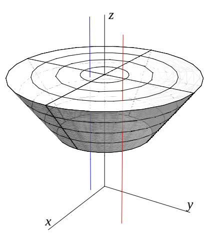
(2) This is true because $z=\sqrt{3(x^2+y^2)}$ can be written in cylindrical coordinates as $z=r\sqrt{3}$. In terms of spherical coordinates, this says that $\rho\cos\phi=\sqrt{3}\rho\sin\phi$. That's the same as saying $\tan\phi=\frac{1}{\sqrt{3}}$, or $\phi=\frac{\pi}{6}$.



We'll use polar coordinates to write the iterated (double) integral describing this slice. The circle can be described as $0 \le \theta < 2\pi$ and $0 \le r \le z$ (and remember that we are still thinking of z as a constant), so the appropriate integral is $\int_0^{2\pi} \int_0^z 1 \cdot r \; dr \; d\theta$.

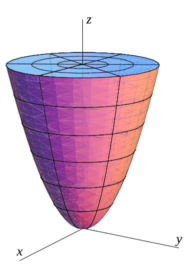
Putting this into our outer integral, we get the iterated integral $\boxed{\int_1^2 \int_0^{2\pi} \int_0^z 1 \cdot r \; dr \; d\theta \; dz}$.

Note: For this problem, writing the inner integral first doesn't work as well, at least not if we want to write the integral with dz as the inner integral. Why? Well, if we try to write the integral with dz as the inner integral, we imagine sticking vertical lines through the solid. The problem is that there are different "types" of vertical lines. For instance, along the red line in the picture below, z goes from the cone ($z = \sqrt{x^2 + y^2}$ or $z = r$) to $z = 2$ (in the solid). But, along the blue line, z goes from $z = 1$ to $z = 2$. So, we'd have to write two separate integrals to deal with these two different situations.



6. *Let $\mathcal{U}$ be the solid enclosed by $z = x^2 + y^2$ and $z = 9$. Rewrite the triple integral $\iiint_{\mathcal{U}} x \; dV$ as an iterated integral. (You need not evaluate, but can you guess what the answer is?)*

Solution. $z = x^2 + y^2$ describes a paraboloid, so the solid looks like this:



Since the solid is symmetric about the z-axis, a good guess is that cylindrical coordinates will make things easier. In cylindrical coordinates, the integrand x is equal to $r \cos\theta$.

Let's think of slicing the solid, which means we'll write our outer integral first. If we slice parallel to the xy-plane, then we are slicing the interval $[0, 9]$ on the z-axis, so our outer integral is $\int_0^9$ something dz.

We use the inner two integrals to describe a typical slice; within a slice, z is constant. Each slice is a disk enclosed by the circle $x^2 + y^2 = z$ (which has radius $\sqrt{z}$). We know that we can describe this in polar coordinates as $0 \le r \le \sqrt{z}$, $0 \le \theta < 2\pi$. So, the inner two integrals will be $\int_0^{2\pi} \int_0^{\sqrt{z}} (r \cos\theta) \cdot r \, dr \, d\theta$. Therefore, the given triple integral is equal to the iterated integral

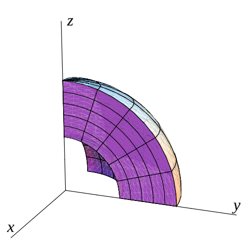
$$\begin{aligned}
\boxed{\int_0^9 \int_0^{2\pi} \int_0^{\sqrt{z}} r\cos\theta \cdot r \, dr \, d\theta \, dz} &= \int_0^9 \int_0^{2\pi} \left(\frac{1}{3} r^3 \cos\theta \Big|_{r=0}^{r=\sqrt{z}} \right) dr \, d\theta \, dz \\
&= \int_0^9 \int_0^{2\pi} \frac{1}{3} z^{3/2} \cos\theta \, d\theta \, dz \\
&= \int_0^9 \left(\frac{1}{3} z^{3/2} \sin\theta \Big|_{\theta=0}^{\theta=2\pi} \right) dz \\
&= \boxed{0}
\end{aligned}$$

That the answer is 0 should not be surprising because the integrand $f(x, y, z) = x$ is anti-symmetric about the plane $x = 0$ (this is sort of like saying the function is odd: $f(-x, y, z) = -f(x, y, z)$), but the solid is symmetric about the plane $x = 0$.

Note: If you decided to do the inner integral first, you probably ended up with dz as your inner integral. In this case, a valid iterated integral is $\boxed{\int_0^{2\pi} \int_0^3 \int_{r^2}^9 r\cos\theta \cdot r \, dz \, dr \, d\theta}$.

7. *The iterated integral in spherical coordinates* $\int_{\pi/2}^{\pi} \int_0^{\pi/2} \int_1^2 \rho^3 \sin^3\phi \, d\rho \, d\phi \, d\theta$ *computes the mass of a solid. Describe the solid (its shape and its density at any point).*

Solution. The shape of the solid is described by the region of integration. We can read this off from the bounds of integration: it is $\frac{\pi}{2} \le \theta \le \pi$, $0 \le \phi \le \frac{\pi}{2}$, $1 \le \rho \le 2$. We can visualize $1 \le \rho \le 2$ by imagining a solid ball of radius 2 with a solid ball of radius 1 taken out of the middle. $0 \le \phi \le \frac{\pi}{2}$ tells us we'll only have the top half of that, and $\frac{\pi}{2} \le \theta \le \pi$ tells us that we'll only be looking at one octant: the one with x negative and y positive:



To figure out the density, remember that we get mass by integrating the density. If we call this solid $\mathcal{U}$, then the iterated integral in the problem is the same as the triple integral $\iiint_{\mathcal{U}} \rho \sin^2\phi \, dV$ since dV is $\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$. So, the density of the solid at a point (ρ, ϕ, θ) is $\boxed{\rho \sin^2\phi}$.