Let $\varepsilon > 0$, $\exists\ \delta > 0$, s.t. $d(x, x_0) = |x - x_0| < \delta$

$\Rightarrow\ d(f(x), f(x_0)) = |f(x) - f(x_0)|$

$= |x - x_0| < \delta = \varepsilon$

$\therefore\ d(f(x), f(x_0)) < \varepsilon$

$\Rightarrow\ f$ is continuous function

Example (5.3): Let (R, d) be usual metric space and $f(x) = \frac{1}{x}$, show that f is continuous at 2 .

Proof:

Let $\varepsilon > 0$, $\exists\ \delta > 0$, s.t.

$d(x, 2) = |x - 2| < \delta \quad \Rightarrow \quad |x| < \delta + 2$

$\Rightarrow\ d(f(x), f(2)) = |f(x) - f(2)|$

$= \left|\frac{1}{x} - \frac{1}{2}\right|$

$= \left|\frac{2-x}{2x}\right|$

$= \frac{|x-2|}{|2x|} < \frac{\delta}{2|x|} < \frac{\delta}{2(\delta+2)} = \frac{\delta}{2\delta+4}$

$\therefore\ \ \varepsilon = \frac{\delta}{2\delta+4}$

$\therefore\ d(f(x), f(2)) < \frac{\delta}{2\delta+4} = \varepsilon$

$\Rightarrow\ f$ is continuous function at 2

Example (5.4): Let (R, d) be usual metric space and $f(x) = \sin x$, show that f is continuous at $x_0 \in R$.

Proof:

Let $\varepsilon > 0$, $\exists\ \delta > 0$, s.t. $d(x, x_0) = |x - x_0| < \delta$

$\Rightarrow\ d(f(x), f(x_0)) = |f(x) - f(x_0)|$

$= |\sin x - \sin x_0|$

$= \left|2 \cos \frac{x-x_0}{2}\ \ \sin \frac{x-x_0}{2}\right|$

$$\leq \left|2\, sin\, \frac{x-x_0}{2}\right|$$

$$\leq 2\left|\frac{x-x_0}{2}\right| \leq |x-x_0| < \delta = \varepsilon$$

$\therefore\ d\big(f(x), f(x_0)\big) < \varepsilon$

$\Rightarrow\ f$ is continuous function at x_0

Remark (5.1): If f is continuous at each point $x \in X$. Then we say that f is continuous on X.

Theorem (5.1): A function $f: X \to Y$ is continuous iff the inverse image of each open set in Y is open in X.

Proof:

Suppose that f is continuous on X

Let $G^* \subset Y$ be any open set

If $x \in f^{-1}(G^*)$ be any point

$\Rightarrow\ f(x) \in G^*$

Since G^* is open set

$\Rightarrow\ f(x)$ is an interior point of G^*

$\Rightarrow\ \exists\ \varepsilon > 0$ s.t. $B_\varepsilon\big(f(x)\big) \subseteq G^*$

Since f is continuous

$\therefore\ \exists\ \delta > 0$, s.t. $f\big(B_\delta(x)\big) \subset B_\varepsilon\big(f(x)\big)$

$\Rightarrow\ B_\delta(x) \subseteq f^{-1}(G^*)$

$\Rightarrow\ x$ is an interior point of $f^{-1}(G^*)$

$\Rightarrow\ f^{-1}(G^*)$ is an open set in X

<u>Conversly</u>, we have $f^{-1}(G^*) \subset X$ is open, $\forall^{open}\ G^* \subset Y$

Let $x \in X$ s.t. $x \in f^{-1}(G^*)$

Since $f^{-1}(G^*)$ is an open set in X

$\Rightarrow\ x$ is an interior point of $f^{-1}(G^*)$

Hence, $\exists\, \delta > 0$, s.t. $B_\delta(x) \subseteq f^{-1}(G^*)$

$\Rightarrow\ f\big(B_\delta(x)\big) \subseteq G^*$

For $\varepsilon > 0$,

$\Rightarrow\ f\big(B_\delta(x)\big) \subseteq B_\varepsilon\big(f(x)\big) \subseteq G^*$

$\Rightarrow\ f$ is continuous on X

Theorem (5.2): A function $f: X \to Y$ is continuous iff the inverse image of each closed set in Y is closed in X.

Proof:

Suppose that f is continuous on X

$\therefore$ The inverse image of each open set in Y is open in X

Let $F^* \subset Y$ be any closed set

$\Rightarrow\ {F^*}^c$ is open set in Y

$\Rightarrow\ f^{-1}\big({F^*}^c\big) = \big(f^{-1}(F^*)\big)^c$ is open

$\Rightarrow\ f^{-1}(F^*)$ is closed in X

Conversely, Let the inverse image of each closed set in Y is closed in X.

Let $G^* \subset Y$ be any open set

$\Rightarrow\ {G^*}^c$ is closed set in Y

$\Rightarrow\ f^{-1}\big({G^*}^c\big) = \big(f^{-1}(G^*)\big)^c$ is closed in X

$\Rightarrow\ f^{-1}(G^*)$ is open in X

$\therefore$ The inverse image of each open set in Y is open in X

$\Rightarrow\ f$ is continuous

Theorem (5.3): A function $f: X \to Y$ is continuous iff $f(\bar{E}) \subset \overline{f(E)}, \forall\, E \subset X$.

Proof:

Suppose that f is continuous on X

$\because\ f(E) \subset \overline{f(E)}$, $\forall\, E \subset X$ and $\overline{f(E)}$ closed

$\Rightarrow\ E \subset f^{-1}(\overline{f(E)})$ closed

$\because$ The inverse image of each closed set in Y is closed in X

$\Rightarrow \bar{E} \subset \overline{f^{-1}\left(\overline{f(E)}\right)} = f^{-1}\left(\overline{f(E)}\right)$

$\Rightarrow \bar{E} \subset f^{-1}\left(\overline{f(E)}\right)$

$\Rightarrow f(\bar{E}) \subset \overline{f(E)}$

Conversely, suppose that $f(\bar{E}) \subset \overline{f(E)}, \forall E \subset X$

Let $F^* \subset Y$ be any closed set

Let $F = f^{-1}(F^*)$

We are given $f(\bar{F}) \subset \overline{f(F)} = \overline{f\left(f^{-1}(F^*)\right)} = \overline{F^*} = F^* = f(F)$

$\Rightarrow f(\bar{F}) \subset f(F)$ but $F \subset \bar{F} \Rightarrow f(F) \subset f(\bar{F})$

$\Rightarrow f(\bar{F}) = f(F)$

$\Rightarrow f^{-1}(f(\bar{F})) = f^{-1}\left(f(F)\right)$

$\Rightarrow \bar{F} = F = f^{-1}(F^*)$

$\Rightarrow f^{-1}(F^*)$ is closed in X

Since the inverse image of each closed set in Y is closed in X

$\Rightarrow f$ is continuous

Theorem (5.4): If $f: X \to Y$ and $g: Y \to Z$ are continuous functions on X and Y respectively, Then $g \circ f: X \to Z$ is continuous on X.

Solution:

Let $G^{**} \subset Z$ be any open set

Since g continuous on Y

$\Rightarrow g^{-1}(G^{**})$ is open in Y

Since f continuous on X

$\Rightarrow f^{-1}(g^{-1}(G^{**}))$ is open in X

$\Rightarrow (f^{-1} \circ g^{-1})(G^{**})$ is open in X

$\Rightarrow (g \circ f)^{-1}(G^{**})$ is open in X

$\Rightarrow (g \circ f)$ is continuous on X
