$$\Delta(G_1) = \sum_{n=1}^{3} \Delta(I_n) = \Delta(I_1) + \Delta(I_2) + \Delta(I_3)$$

$$= (1 - (-2)) + (3 - 1) + (6 - 4) = 7$$

$$\Delta(G_2) = \sum_{n=1}^{3} \Delta(I_n) = \Delta(I_1) + \Delta(I_2) + \Delta(I_3)$$

$$= (2 - 0) + (5 - 3) + (7 - 5) = 6$$

$$\Delta(G_3) = \sum_{n=1}^{3} \Delta(I_n) = \Delta(I_1) + \Delta(I_2) + \Delta(I_3)$$

$$= (3 - 1) + (4 - 3) + (6 - 5) = 4$$

We have

$G_1 \cup G_2 \cup G_3 = \{(-2,3) \cup (3,7)\}$,

$$\Delta\left(\bigcup_{n=1}^{3} G_n\right) = \Delta(G_1 \cup G_2 \cup G_3)$$

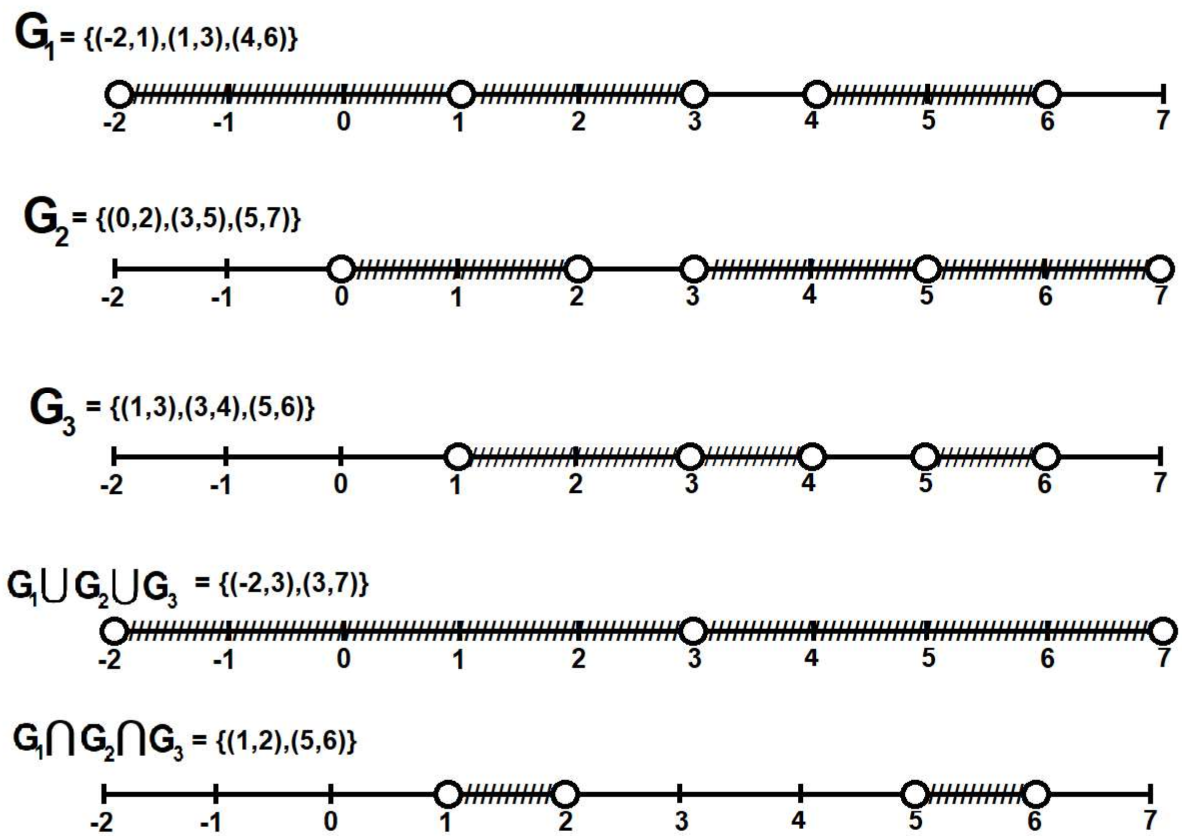
$$= (3 - (-2)) + (7 - 3) = 9$$

and

$$\sum_{n=1}^{3} \Delta(G_n) = \Delta(G_1) + \Delta(G_2) + \Delta(G_3) = 7 + 6 + 4 = 17$$

Note that

$$\Delta\left(\bigcup_{n=1}^{3} G_n\right) = 9 < \sum_{n=1}^{3} \Delta(G_n) = 17$$



Definition (3.4): (Outer Measure) القياس الخارجي

If $S \subset [a, b]$, then $\overline{\mu}(S)$, called the **outer measure** of S, is defined as

$$\overline{\mu}(S) = inf\{\Delta(G) : \forall \ open\ set\ G \supseteq S\},$$

Note (3.3):

(1) If G is a bounded open set, then $\overline{\mu}(G) = \Delta(G)$.

(2) $\overline{\mu}(\emptyset) = \Delta(\emptyset) = 0$.

Example (3.4): Let $S = \{(0,1) \cup (1,2) \cup (3,5)\}$, find the outer measure of S.

Solution:

$$\overline{\mu}(S) = \sum_{n=1}^{3} \Delta(I_n) = \Delta(I_1) + \Delta(I_2) + \Delta(I_3)$$

$$= (1-0) + (2-1) + (5-3) = 4$$

Example (3.5): Let $S = \{x\}$, find $\overline{\mu}(S)$.

Solution:

Let $I_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right), \forall\, n \geq 1,$

Clear that

$S \subseteq I_n$ and $\Delta I_n = x + \frac{1}{n} - \left(x - \frac{1}{n}\right) = \frac{2}{n}$

$\Rightarrow\ \overline{\mu}(S) = inf\left\{\frac{2}{n}\right\} = 0$

$\therefore\ \overline{\mu}(S) = 0$.

Example (3.6): Let S be a countable set (finite or infinite), find $\overline{\mu}(S)$.

Solution:

Let $S = \{x_1, x_2, \dots, x_n, \dots\}$, and

let $I_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right),\ \ \forall\ \varepsilon \geq 0\,,\ \ \forall\ n \geq 1.$

Clear that

$x_1 \in I_1 = \left(x_1 - \frac{\varepsilon}{4}, x_1 + \frac{\varepsilon}{4}\right),$

$x_2 \in I_2 = \left(x_2 - \frac{\varepsilon}{8}, x_2 + \frac{\varepsilon}{8}\right)$,

$\vdots$

$$x_n \in I_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right)$$

and $S \subseteq \bigcup_n I_n$

Now,

$$\Delta(\bigcup_n I_n) = \sum_{n=1}^{\infty} \Delta(I_n)$$

$$= \sum_{n=1}^{\infty} \left(x_n + \frac{\varepsilon}{2^{n+1}} - \left(x_n - \frac{\varepsilon}{2^{n+1}} \right) \right)$$

$$= \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$$

$$= \varepsilon \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right) \qquad (infinite\ geometric\ series)$$

$$= \varepsilon \left(\frac{\frac{1}{2}}{1 - \frac{1}{2}}\right) = \varepsilon$$

$\therefore\ \overline{\mu}(S) \leq inf\{\Delta(\bigcup_n I_n)\} = inf\{\varepsilon\} = 0.$

Example (3.7): Let $S = [a, b]$, find the outer measure of S.

Solution:

Let $I_n = \left(a - \frac{1}{n}, b + \frac{1}{n}\right)$, $\forall\ n \geq 1$

Clear that

$S \subseteq\ I_n$ and $\Delta(I_n) = b + \frac{1}{n} - \left(a - \frac{1}{n}\right) = b - a + \frac{2}{n}$

$\therefore\ \overline{\mu}(S) = inf\{\Delta(I_n)\}$

$$= inf\left\{b - a + \frac{2}{n}\right\} = b - a.$$

Theorem (3.3): (Properties of Outer Measure)

Let $S_1, S_2, \dots, S_n$ be bounded sets, then

(1) $\overline{\mu}(S) \geq 0.$