

Triple integrals التكاملات الثلاثية

Definition: Let D be the region defined by the inequalities $a \leq x \leq b$, $h_1(x) \leq y \leq h_2(x)$, $g_1(x, y) \leq z \leq g_2(x, y)$. If $f(x, y, z)$ is continuous function of three variables, then the triple integral is

$$V = \iiint_D f(x, y, z) dV = \int_a^b \int_{h_1(x)}^{h_2(x)} \int_{g_1(x, y)}^{g_2(x, y)} dz dy dx$$

Definition: The volume of a closed, bounded region D in space is

$$V = \iiint_D dV$$

Example 1: Find the value of triple integral for the function

$f(x, y, z) = 2xyz$ and bounded by surface $z = 2 - \frac{1}{2}x^2$ and the planes $z = 0$, $y = x$, $y = 0$.

Solution:

$$\begin{aligned} V &= \iiint_D 2xyz dV = \int_0^2 \int_0^x \int_0^{2-\frac{1}{2}x^2} 2xyz dz dy dx \\ &= \int_0^2 \int_0^x xyz^2 \Big|_0^{2-\frac{1}{2}x^2} dy dx \\ &= \int_0^2 \int_0^x xy(2 - \frac{1}{2}x^2)^2 dy dx = \int_0^2 \int_0^x xy \left[4 - 2x^2 + \frac{1}{4}x^4 \right] dy dx = \\ &= \int_0^2 \int_0^x \left[4xy - 2x^3y + \frac{1}{4}x^5y \right] dy dx = \int_0^2 \left(2xy^2 - x^3y^2 + \frac{1}{8}x^5y^2 \right) \Big|_0^x dx \\ &= \int_0^2 \left[2x^3 - x^5 + \frac{1}{8}x^7 \right] dx = \left[\frac{x^4}{2} - \frac{x^6}{6} + \frac{1}{64}x^8 \right]_0^2 = 8 - \frac{64}{6} + \frac{256}{64} = \frac{4}{3} \end{aligned}$$

Example 2: Use **triple integral** to find the volume in the first octant (الثلث) of the region that bounded by two surfaces $x^2 + 4y^2 = 4z$ and $x^2 + 4y^2 = 48 - 8z$.

Solution:

$$\begin{aligned} V &= \iiint_D dV = \int_0^2 \int_0^{2\sqrt{4-y^2}} \int_{\frac{x^2+4y^2}{4}}^{\frac{48-x^2-4y^2}{8}} dz \, dx \, dy \\ &= \int_0^2 \int_0^{2\sqrt{4-y^2}} \left[\frac{48-x^2-4y^2}{8} - \frac{x^2+4y^2}{4} \right] dx \, dy = \int_0^2 \int_0^{2\sqrt{4-y^2}} \frac{3}{8} (16 - x^2 - 4y^2) dx \, dy \end{aligned}$$

Example 3 Use **triple integral** to find the volume in the first octant (الثلث) of the region that bounded by the surface $2z = 4 - x^2 - y^2$ and the plane xy .

Solution:

$$V = \iiint_D dV = \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{\frac{4-x^2-y^2}{2}} dz \, dy \, dx$$

Example 4: Let D be the solid region that bounded above sphere (الكرة) $x^2 + y^2 + z^2 = 1$ and below $z^2 = x^2 + y^2$, find the value

$$V = \iiint_D z \, dV.$$

Solution:

$$V = \iiint_D z \, dV = \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \int_{-\sqrt{\frac{1}{2}-x^2}}^{\sqrt{\frac{1}{2}-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z \, dz \, dy \, dx$$

Example 5: Use **triple integral** to find the volume the solid within the cylinder $x^2 + y^2 = 9$ and between the planes $z = 1$ and $x + z = 5$.

Solution:

$$\begin{aligned} V &= \iint_R \int_1^{5-x} dz \, dA = \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_1^{5-x} dz \, dx dy \\ &= \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} z \Big|_1^{5-x} dx \, dy = \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (4-x) dx \, dy = 4 \int_0^3 \int_0^{\sqrt{9-y^2}} (4-x) dx \, dy = \\ &= 4 \int_0^3 \left(4x - \frac{x^2}{2} \right) \Big|_0^{\sqrt{9-y^2}} dy = 4 \int_0^3 \left(4\sqrt{9-y^2} - \frac{9-y^2}{2} \right) dy \end{aligned}$$

Example 6: Use **triple integral** to find the volume of the solid enclosed between the xy- plane and $3z = 9 - x^2 - y^2$.

Solution:

$$\begin{aligned} \iint_R \int_0^{\frac{9-x^2-y^2}{3}} dz \, dA &= \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} z \Big|_0^{\frac{9-x^2-y^2}{3}} dx \, dy = \\ \frac{1}{3} \int_{-3}^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} (9 - x^2 - y^2) dx \, dy &= \frac{4}{3} \int_0^3 \int_0^{\sqrt{9-y^2}} (9 - x^2 - y^2) dx \, dy = \end{aligned}$$

We can use polar coordinates

$$\begin{aligned} &= \frac{4}{3} \int_0^{\frac{\pi}{2}} \int_0^3 (9 - r^2) r dr \, d\theta = -\frac{2}{3} \int_0^{\frac{\pi}{2}} (9 - r^2)^2 \Big|_0^3 d\theta = (27) \int_0^{\frac{\pi}{2}} d\theta = (27) \theta \Big|_0^{\frac{\pi}{2}} \\ &= \frac{27}{2} \pi \end{aligned}$$

Example 7: Use **triple integral** to find the volume bounded above surfaces

$$z = 8 - x^2 - y^2 \text{ and below } z = x^2 + 3y^2 .$$

Solution:

$$8 - x^2 - y^2 = x^2 + 3y^2 \Rightarrow 4y^2 = 8 - 2x^2 \Rightarrow 2y^2 = 4 - x^2 \Rightarrow y = \pm \sqrt{\frac{4-x^2}{2}}$$

$$\int_{-2}^2 \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} dz \, dy \, dx = \int_{-2}^2 \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} z \Big|_{x^2+3y^2}^{8-x^2-y^2} dy \, dx$$

$$= \int_{-2}^2 \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8 - x^2 - y^2 - (x^2 + 3y^2)) dy \, dx = \int_{-2}^2 \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (8 - 2x^2 + 4y^2) dy \, dx$$

$$= \int_{-2}^2 \left(8y - 2x^2y + \frac{4y^3}{3} \right) \Big|_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} dx$$

$$= \int_{-2}^2 \left(8\sqrt{\frac{4-x^2}{2}} - 2x^2\sqrt{\frac{4-x^2}{2}} + \frac{4(\sqrt{\frac{4-x^2}{2}})^3}{3} \right) - \left(-8\sqrt{\frac{4-x^2}{2}} + 2x^2\sqrt{\frac{4-x^2}{2}} - \frac{4(\sqrt{\frac{4-x^2}{2}})^3}{3} \right) dx =$$

$$\int_{-2}^2 \left(16\sqrt{\frac{4-x^2}{2}} - 4x^2\sqrt{\frac{4-x^2}{2}} + \frac{8\sqrt{\frac{4-x^2}{2}}^3}{3} \right) dx =$$

$$\int_{-2}^2 \left(16\sqrt{\frac{4-x^2}{2}} - 4x^2\sqrt{\frac{4-x^2}{2}} + \frac{8\sqrt{\frac{4-x^2}{2}}^3}{3} \right) dx = \frac{4\sqrt{2}}{3} \int_{-2}^2 (4-x^2)^{\frac{3}{2}} dx = 8\sqrt{2}\pi$$