12th Lecture

Remark (4.1): If X is compact, then we say that (X,τ) is compact topological space.

Example (4.3): If (X,τ) is the indiscrete topological space. Then X is compact.

Theorem (4.2): (Heine – Borel Theorem) نظرية هاين - بوريل

A subset of R is compact iff it is closed and bounded.

Remark (4.2): Heine – Borel Theorem does not true in the topological space.

Theorem (4.3): Every closed subset of a compact topological space (X,τ) is compact.

Proof: Let $E \subset X$ be closed

Let $\{G_\alpha\}_{\alpha\in\Lambda}$ be an open cover of $E \Rightarrow E \subset \bigcup_{\alpha\in\Lambda} G_\alpha$

Since E closed $\Rightarrow E^c$ open and

Since $E \cup E^c = X \Rightarrow E \cup E^c \subset (\bigcup_{\alpha\in\Lambda} G_\alpha) \cup E^c$

$\Rightarrow X \subset \bigcup_{\alpha\in\Lambda} (G_\alpha \cup E^c)$

$\Rightarrow \{G_\alpha \cup E^c)\}_{\alpha\in\Lambda}$ is an open cover of X (compact)

$\Rightarrow X \subset \bigcup_{i=1}^{n} G_{\alpha_i} \cup E^c$

We have $E \subset X$

$\Rightarrow E \subset \bigcup_{i=1}^{n} G_{\alpha_i} \cup E^c \Rightarrow E \subset \bigcup_{i=1}^{n} G_{\alpha_i}$ because $E \cap E^c = \emptyset$

$\Rightarrow E$ is compact.

Definition (4.3): (Finite Intersection Property (F.I.P.)) خاصية التقاطع المنتهي

We say that a family of subsets of X in (X,τ) satisfies the **finite intersection property (F.I.P.)** iff any finite subfamily of it has a nonempty intersection.

In other words:

The family $\{E_\alpha\}_{\alpha\in\Lambda}$ satisfies F.I.P. if $\bigcap_{i=1}^{n} E_{\alpha_i} \neq \emptyset$ for the subfamily $\{E_{\alpha_i}\}_{i=1}^{n}$.

Examples (4.4):

(1) The family $\left\{\left(0,\frac{1}{n}\right)\right\}_{n\in N}$ satisfies F.I.P. because $\bigcap_{i=1}^{n}(0,\frac{1}{n}) \neq \emptyset$ while $\bigcap\limits_{\alpha\in\Lambda}(0,\frac{1}{n}) = \emptyset$.

(2) The family $\{I_n\}_{n\in Z}$ where $I_n = (-\infty, n]$ of closed sets satisfies the F.I.P. because $\bigcap\limits_{n\in\{-10,-9,\ldots,0,1,2,\ldots,N\}}(-\infty,n] = (-\infty,-10] \neq \emptyset$

while $\bigcap\limits_{n\in Z}(-\infty,n] = (-\infty,-\infty) = \emptyset$.

Theorem (4.4): If (X,τ) is a compact topological space. Then every family of closed sets in X satisfying the F.I.P. has a nonempty intersection.

Proof: Let (X,τ) be a compact

Let $\{F_\alpha\}_{\alpha\in\Lambda}$ be a family of closed sets in X satisfying F.I.P. (i.e. $\bigcap_{i=1}^{n} F_{\alpha_i} \neq \emptyset$).

We need to prove that $\bigcap\limits_{\alpha\in\Lambda} F_\alpha \neq \emptyset$

Assume that $\bigcap\limits_{\alpha\in\Lambda} F_\alpha = \emptyset$

$\Rightarrow\ \emptyset^c = (\bigcap\limits_{\alpha\in\Lambda} F_\alpha)^c \Rightarrow\ X = \bigcup\limits_{\alpha\in\Lambda} F_\alpha^c$

Since F_α^c is open $\forall\ \alpha \in \Lambda$

$\Rightarrow\ \{F_\alpha^c\}_{\alpha\in\Lambda}$ is an open cover of X (X compact)

$\Rightarrow\ \{F_{\alpha_i}^c\}_{i=1}^{n}$ is a finite subcover of X

$\Rightarrow\ X = \bigcup_{i=1}^{n} F_{\alpha_i}^c \ \Rightarrow\ X^c = \left(\bigcup_{i=1}^{n} F_{\alpha_i}^c\right)^c$

$\Rightarrow\ \emptyset = \bigcap_{i=1}^{n} F_{\alpha_i} \quad \Rightarrow$ Contradiction

Hence $\bigcap\limits_{\alpha\in\Lambda} F_\alpha \neq \emptyset$

Theorem (4.5): If a family of closed sets in (X,τ) satisfying the F.I.P. has a nonempty intersection. Then (X,τ) is compact.

Proof: Let (X,τ) be a topological space

Let $\{F_\alpha\}_{\alpha\in\Lambda}$ be a family of closed sets in (X,τ) satisfying F.I.P. and $\bigcap\limits_{\alpha\in\Lambda} F_\alpha \neq \emptyset$

We need to prove that (X,τ) is compact

Assume that (X,τ) is not compact

$\Rightarrow\ \exists$ open cover of X which has no finite subcover of X

$\Rightarrow\ X = \bigcup\limits_{\alpha\in\Lambda} G_\alpha$ but $X \neq \bigcup_{i=1}^{n} G_{\alpha_i}$

$\Rightarrow\ X^c = (\bigcup\limits_{\alpha\in\Lambda} G_\alpha)^c$ but $X^c \neq \left(\bigcup_{i=1}^{n} G_{\alpha_i}\right)^c$

$\Rightarrow\ \bigcap\limits_{\alpha\in\Lambda} G_\alpha^c = \emptyset$ but $\bigcap_{i=1}^{n} G_{\alpha_i}^c \neq \emptyset$

$\Rightarrow$ The family of $\{G_\alpha^c\}_{\alpha\in\Lambda}$ of closed sets in X satisfies F.I.P. but $\bigcap\limits_{\alpha\in\Lambda} G_\alpha^c = \emptyset$

$\Rightarrow$ Contradiction $\Rightarrow$ (X,τ) is compact.

Definition (4.4): (Sequentially Compactness) **التـراص التـتـابـعـي**

We say that a subset E of a topological space (X,τ) is **sequentially compact** iff every sequence of E contains a subsequence converges to a point in E.

Symbolically:

($E \subset X$ is sequentially compact) $\Leftrightarrow$

($\forall^{\text{sequence}}\ \{a_n\}_{n\in N}\ in\ E,\ \exists^{\text{subsequence}}\ \{a_{n_k}\}$ converge to $x \in E$)

Examples (4.5): Let $E = (0,1)$ is in (R, τ) usual topology on R, E is not sequentially compact because $\{a_n\}_{n\in N} = \left\{\frac{1}{n+1}\right\}_{n\in N}$ is a sequence in E, $0 < \frac{1}{n+1} < 1$, while every subsequence of $\{a_n\}_{n\in N}$ converges to $0 \notin E$.

For example: The subsequence "$\left\{\frac{1}{(2n)^2}\right\}_{n\in N}$" converges to $0 \notin E$.

Definition (4.5): (Countably Compactness) **التـراص الـعّدي**

We say that a subset E of a topological space (X, τ) is **countably compact** iff every infinite subset $A \subset E$ has a limit point x in E.

Symbolically:

($E \subset X$ is countably compact) ⇔ ($\forall^{\text{infinite}}\ A \subset E, A$ has a limit point $x \in E$)

Examples (4.6): The sets $E = (0,1]$, $E = (0,1)$, $E = [0,1)$ are not countably compact in (R, τ).

Solution: Take $E = (0,1]$

Let $A = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots, \frac{n}{n+1}, \frac{n+1}{n+2}, \dots\} \subset E$

$d(A) = \{1\}$ and $E = (0,1]$

Now $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n+1}\} \subset E$

$d(A) = 0 \ \wedge \ 0 \notin E$, Then E is not countably compact

Now, $E = (0,1]$

Let $A = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$, We have $A \subset E$ and infinite

The limit of A is $0 \notin E$,

$\therefore$ E is not countably compact

Now, $E = [0,1)$

$d(A) = 1 \ \wedge \ 1 \notin E$

$\therefore$ E is not countably compact.