Q1) prove that $|z_1 z_2| = r_1 r_2$, $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$.

proof:- let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ $(\cos\theta_2 + i\sin\theta_2)$

$$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 = r_1 r_2 \,(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$$

$$= r_1 r_2\,(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

$$|z_1 z_2| = |r_1 r_2|\,|\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)|$$

$$= |r_1 r_2|\sqrt{\cos^2(\theta_1+\theta_2) + \sin^2(\theta_1+\theta_2)}$$

$$= r_1 r_2\sqrt{1}$$

$$= r_1 r_2$$

$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2 .$$

Q2) prove that $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2}$, $\arg\frac{z_1}{z_2} = \arg(z_1) - \arg(z_2)$.

proof:- let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\,\frac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} \times \frac{\cos\theta_2 - i\sin\theta_2}{\cos\theta_2 - i\sin\theta_2}$$

$$= \frac{r_1}{r_2}\,\frac{\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 - i\sin\theta_2\cos\theta_1 + \sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}\,\frac{\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)}{1}$$

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{r_1}{r_2}\right|\sqrt{\cos^2(\theta_1-\theta_2) + \sin^2(\theta_1-\theta_2)} = \frac{|r_1|}{|r_2|}\sqrt{1} = \frac{r_1}{r_2} .$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2 .$$

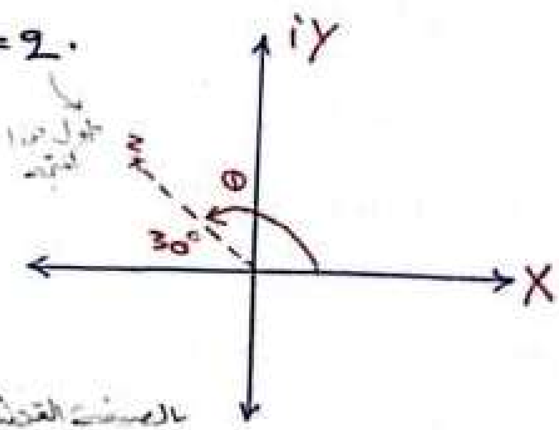
DeMoivre's theorem:-

let $Z \in \mathbb{C}$ and $Z = r(\cos\theta + i\sin\theta)$, then there exist $n \in Z^+$ such that $Z^n = [r(\cos\theta + i\sin\theta)]^n$

$$= r^n(\cos n\theta + i\sin n\theta).$$

Ex.:- Find 1. $(-\sqrt{3}+i)^7$ 2. $(1+i)^{11}$. 3. $(0.4)^4$

Sol.(1):- $r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{4} = 2.$



$$\tan\theta = \frac{1}{-\sqrt{3}}$$

$\therefore \theta = 180 - 30 = 150°$

or $\theta = \pi - \frac{\pi}{6}$

$Z = 2(\cos 150 + i\sin 150)$ الصيغة القطبية

$Z^7 = 2^7(\cos 7(150) + i\sin 7(150)) = 64\sqrt{3} - 64i$

Sol.(2):- $r = \sqrt{1+1} = \sqrt{2}$

$\tan\theta = \frac{1}{1} = 1$, $\theta = \frac{\pi}{4}$

$Z = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$ $\therefore Z^{11} =$

EX(3):- Find $(0,2)^4$ $\quad r=\sqrt{0^2+2^2}=\sqrt{4}=2$

Sol.:- $r=2$, $\theta=\frac{\pi}{2}=90^\circ$

بالربع الاول

$$Z^4=2^4\left(\cos 4\left(\tfrac{\pi}{2}\right)+i\sin 4\left(\tfrac{\pi}{2}\right)\right)$$

$$=2^4(\cos 2\pi+i\sin 2\pi).$$

EX.:- prove that $\cos 2\theta=\cos^2\theta-\sin^2\theta$.

$\sin 2\theta=2\cos\theta\sin\theta$.

Proof:- by DeMoveres Theorem.

$$(\cos\theta+i\sin\theta)^2=\cos 2\theta+i\sin 2\theta.$$

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and $(\cos\theta+i\sin\theta)^2=(\cos\theta+i\sin\theta)(\cos\theta+i\sin\theta)$.

$$=\cos\theta\cos\theta+i\sin\theta\cos\theta+i\sin\theta\cos\theta-\sin\theta\sin\theta.$$

$$=\cos^2\theta-\sin^2\theta+2i\sin\theta\cos\theta.$$

$$\therefore \cos 2\theta+i\sin 2\theta=\cos^2\theta-\sin^2\theta+2i\cos\theta\sin\theta.$$

$\therefore \cos 2\theta=\cos^2\theta-\sin^2\theta$, $\quad$ حقيقي = حقيقي

$\sin 2\theta=2\cos\theta\sin\theta$. $\quad$ خيالي = خيالي

H.W

prove that $\cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$.

$\sin 3\theta=3\cos^2\theta\sin\theta-\sin^3\theta$.

The roots of complex numbers:-

If $w, z \in \mathbb{C}$ with $w^n = z$, then w is called the n.th root of z.

Theorem:- for all complex numbers is not equal to zero with $z = r(\cos\theta + i\sin\theta)$ has n of n.th root.

i.e. $$w_k = \sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right), \quad k = 0, 1, 2, \dots n-1$$

Ex.:- Find the roots of the equation $z^3 = 8i$.

$z^3 = 8i \quad , a = 0 \quad , b = 8 \quad , n = 3$

$r = \sqrt{a^2 + b^2} = 8 = 2^3$

$\theta = \tan^{-1}\frac{8}{0} \quad , \theta = 90^\circ$

$$w_k = \sqrt[3]{8}\left(\cos\frac{90 + 2\pi k}{3} + i\sin\frac{90 + 2\pi k}{3}\right), \quad k = 0, 1, 2$$

when k=0

$w_0 = 2\left(\cos\frac{90}{3} + i\sin\frac{90}{3}\right)$

$= 2(\cos 30 + i\sin 30)$

$= 2\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)$

$= \sqrt{3} + i.$

$$W_1 = 2\left(\cos\frac{90+2\pi}{3} + i\sin\frac{90+2\pi}{3}\right) \quad , \; k=1$$

$$= 2\left(\cos\frac{\cancel{3}(30+120)}{\cancel{3}} + i\sin\frac{\cancel{3}(30+120)}{\cancel{3}}\right)$$

$$= 2\left(\cos 150^\circ + i\sin 150^\circ\right)$$

$$= 2\left(\frac{-\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= -\sqrt{3} + i$$

$$W_2 = 2\left(\cos\frac{90+4\pi}{3} + i\sin\frac{90+4\pi}{3}\right) \quad , \; k=2$$

$$= 2\left(\cos\frac{\cancel{3}(30+240)}{\cancel{3}} + i\sin\frac{\cancel{3}(30+240)}{\cancel{3}}\right)$$

$$= 2\left(\cos 270^\circ + i\sin 270^\circ\right)$$

$$= 2\left(0 + (-1)i\right) = 0 - 2i.$$

EX.1- Find $\sqrt[3]{27i}$

$$Z = \sqrt[3]{27i} \Rightarrow Z^3 = 27i$$

$$a=0 \quad , \; b=27 \quad , \; n=3$$

$$r = \sqrt{a^2+b^2} = \sqrt{0^2+27^2} = 27 = 3^3$$

$$\theta = \tan^{-1}\frac{b}{a} = \tan^{-1}\frac{27}{0}$$

$$\Rightarrow \theta = 90 \quad , \; k=0,1,2$$