Theorem (5.5): The continuous image of a compact set is compact.

Proof:

Let $f: X \to Y$ be a continuous function and let $E \subset X$ be compact

We need to show that $E^* = f(E)$ is compact

Let $\{G_\alpha^*\}$ be an open cover of $f(E) = E^*$

$\Rightarrow \; E^* \subset \bigcup_{\alpha\in\Lambda} G_\alpha^*$

$\Rightarrow \; f(E) \subset \bigcup_{\alpha\in\Lambda} G_\alpha^*$

$\Rightarrow \; E \subset f^{-1}(\bigcup_{\alpha\in\Lambda} G_\alpha^*)$

$\Rightarrow \; E \subset \bigcup_{\alpha\in\Lambda} f^{-1}(G_\alpha^*)$

Since $f^{-1}(G_\alpha^*)$ is open $\forall\, \alpha \in \Lambda$

$\Rightarrow \; \{f^{-1}(G_\alpha^*)\}_{\alpha\in\Lambda}$ is an open cover of E (compact)

$\Rightarrow \; E \subset \bigcup_{i=1}^{n} f^{-1}\big(G_{\alpha_i}^*\big)$

$\Rightarrow \; E \subset f^{-1}(\bigcup_{i=1}^{n} G_{\alpha_i}^*)$

$\Rightarrow \; f(E) \subset \bigcup_{i=1}^{n} G_{\alpha_i}^*$

$\Rightarrow \; E^* \subset \bigcup_{i=1}^{n} G_{\alpha_i}^*$

$\Rightarrow \; E^*$ is compact

Definition (5.2): (Bounded Function) **الدالة المقيدة**

The function $f: X \to R$ is called **bounded** if $\exists\, M > 0$ such that $|f(x)| \leq M$, $\forall\, x \in X$.

Example (5.5):

Determine whether the function $f: R \to R$ defined by $f(x) = sin(x)$ is bounded or not.

Solution: Since $|sin(x)| \leq 1 \;,\; \forall\; x \in R$

$\therefore\; f$ is bounded function.

Theorem (5.6): Let X be a compact space and $f: X \to R$ be a continuous function, then f is bounded.

Proof:

Since X is compact and since f is continuous function

$\Rightarrow$ $f(X)$ is compact in R

$\Rightarrow$ $f(X)$ is bounded. (By Heine – Borel Theorem)

Definition (5.3):

Let $f: X \to R$, then we say that x_0 is a **maximum** of f if $f(x_0) \geq f(x)$, $\forall\, x \in X$, and y_0 is a **minimum** of f if $f(y_0) \leq f(x), \forall\, x \in X$.

Example (5.6): Let $f: [0,2] \to R$, defined by $f(x) = x^2$, find the maximum and minimum of f.

Solution:

Since $f(2) = 2^2 = 4 \geq f(x), \forall\; x \in [0,2]$

$\Rightarrow$ $x = 2$ is the maximum of f

Since $f(0) = 0^2 = 0 \leq f(x), \forall\; x \in [0,2]$

$\Rightarrow$ $x = 0$ is the minimum of f

Theorem (5.7): Let $f: X \to R$ be a continuous function and X is a compact, then $\exists\, x_0, y_0 \in X$ such that $f(y_0) \leq f(x) < f(x_0)$.

Definition (5.4): (Uniform Continuous) الاستمرارية المنتظمة

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f: X \to Y$ is **uniformly continuous** if $\forall\, \varepsilon > 0, \exists\, \delta = \delta(\varepsilon) > 0$ such that

$\forall\, x, x_0 \in X$, if $d_X(x, x_0) < \delta \quad \Rightarrow \quad d_Y\big(f(x), f(x_0)\big) < \varepsilon$.

Example (5.7): Show that the function $f(x) = x$ is uniformly continuous in (R, d).

Solution: Let $\varepsilon > 0$, $\exists\ \delta > 0$, s.t.

$d(x, x_0) = |x - x_0| < \delta$

$\Rightarrow\ d\big(f(x), f(x_0)\big) = |f(x) - f(x_0)|$

$= |x - x_0| < \delta$

Since $\varepsilon = \delta$

$\Rightarrow\ f$ is uniformly continuous function.

Example (5.8): Is the function $f(x) = \frac{x^2+1}{2}$ uniformly continuous in (R, d) or not?

Solution:

Let $\varepsilon > 0$, $\exists\ \delta > 0$, s.t.

$d(x, x_0) = |x - x_0| < \delta$

$\Rightarrow\ d\big(f(x), f(x_0)\big) = |f(x) - f(x_0)|$

$$= \left|\frac{x^2+1}{2} - \frac{x_0^2+1}{2}\right|$$

$$= \left|\frac{x^2+1-x_0^2-1}{2}\right|$$

$$= \left|\frac{x^2-x_0^2}{2}\right|$$

$$= \left|\frac{(x+x_0)(x-x_0)}{2}\right|$$

$$= \frac{1}{2}|x + x_0 + x_0 - x_0|\ |x - x_0|$$

$$< \frac{\delta}{2}|x - x_0 + 2x_0| < \frac{\delta}{2}(\delta + 2|x_0|)$$

Since $\varepsilon = \frac{\delta}{2}(\delta + 2|x_0|)$

$\Rightarrow\ f$ is not uniformly continuous function.

Example (5.9): Is the function $f(x) = x^2$ uniformly continuous in (R, d) or not?

Solution:

Let $\varepsilon > 0$, $\exists\, \delta > 0$, s.t.

$d(x, x_0) = |x - x_0| < \delta$

$\Rightarrow\ d\big(f(x), f(x_0)\big) = |f(x) - f(x_0)|$

$= |x^2 - x_0^2|$

$= |(x + x_0)(x - x_0)|$

$= |x + x_0||x - x_0|$

$= \delta|x - x_0 + x_0 + x_0|$

$< \delta|x - x_0 + 2x_0| < \delta(\delta + 2|x_0|)$

Since $\varepsilon = \delta(\delta + 2|x_0|)$

$\Rightarrow\ f$ is not uniformly continuous function.

Example (5.10): Show that $f(x) = c$, (c is constant) uniformly continuous in (R, d).

Solution:

Let $\varepsilon > 0$, $\exists\, \delta > 0$, s.t.

$d(x, x_0) = |x - x_0| < \delta$

$\Rightarrow\ d\big(f(x), f(x_0)\big) = |f(x) - f(x_0)|$

$= |c - c| = 0 < \delta$

Since $\varepsilon = \delta$

$\Rightarrow\ f$ is uniformly continuous function.

Theorem (5.8): Every continuous function in a compact space is uniformly continuous.
