(2) If $S_1 \subseteq S_2$, then $\overline{\mu}(S_1) \leq \overline{\mu}(S_2)$.

(3) $\overline{\mu}\left(\bigcup_n S_n\right) \leq \sum_n^{\infty} \overline{\mu}(S_n)$.

Proof:

(1) Since $\overline{\mu}(S) = inf\{\Delta(G) : \forall \ open\ set\ G \supseteq S\}$,

and $\Delta(G) \geq 0$, $\forall$ open set G

$\Rightarrow in\,f\{\Delta(G)\} \geq 0, \ \ \forall\, G$

$\Rightarrow \overline{\mu}(S) \geq 0$.

(2) We have

$\overline{\mu}(S_1) = inf\{\Delta(G) : \forall \ open\ set\ G \supseteq S_1\}$,

$\overline{\mu}(S_2) = inf\{\Delta(H) : \forall \ open\ set\ H \supseteq S_2\}$

Since $S_1 \subseteq S_2$

$\Rightarrow S_1 \subseteq G \subseteq S_2 \subseteq H, \ \ \forall\, G, H$

$\Rightarrow G \subseteq H$,

$\Rightarrow \Delta(G) \leq \Delta(H)$,

$\Rightarrow in\,f\{\Delta(G)\} \leq in\,f\{\Delta(H)\}, \ \ \forall\, G, H$

$\therefore \ \overline{\mu}(S_1) \leq \overline{\mu}(S_2)$.

(3) $\forall\, \varepsilon > 0, \ \forall\, n \geq 1$, by the definition of outer measure

$\exists$ a bounded open set G_n such that $S_n \subseteq G_n$ and

$\Delta(G_n) - \overline{\mu}(S_n) \leq \frac{\varepsilon}{2^n}$

$\Rightarrow \ \Delta(G_n) \leq \overline{\mu}(S_n) + \frac{\varepsilon}{2^n}$

Let $G = \bigcup_n G_n$

$$\Rightarrow \bigcup_n S_n \subseteq \bigcup_n G_n = G$$

$$\therefore \ \overline{\mu}(S_n) \leq \Delta(G) \leq \sum_{n=1}^{\infty} \Delta(G_n)$$

$$\leq \sum_{n=1}^{\infty} \overline{\mu}(S_n) + \frac{\varepsilon}{2^n}$$

$$= \sum_{n=1}^{\infty} \overline{\mu}(S_n) + \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$$

$$= \sum_{n=1}^{\infty} \overline{\mu}(S_n) + \varepsilon$$

Since ε small value close to zero

$$\therefore \ \overline{\mu}\left(\bigcup_n S_n\right) \leq \sum_n \overline{\mu}(S_n).$$

Exercises (3.1): (Homework)

(1) Let $S_1, S_2 \subseteq R$. Prove that if $\overline{\mu}(S_1) = 0$ then $\overline{\mu}(S_1 \cup S_2) = \overline{\mu}(S_2)$.

(2) Prove that $\overline{\mu}([0,2] \cup [7,11]) = 6$.

Definition (3.5): Let $G, H \subseteq R$, then the **symmetric difference** between G and H is defined by

$$|G - H| = (G - H) \cup (H - G)$$

Definition (3.6): Let $S \subseteq R$ be a bounded set, if $\forall\, \varepsilon > 0,\ \exists$ a bounded open set G such that $\overline{\mu}(|G - S|) < \varepsilon$, then S is **measurable** set and the **measure**

$$\mu(S) = \overline{\mu}(S).$$

Lemma (3.1): The bounded set $S \subseteq R$ is measureable **iff** there exists a bounded open set G such that

$$S \subseteq G \quad \text{and} \quad \overline{\mu}(G - S) < \varepsilon$$

Example (3.8): Let $S = [a, b]$, Prove that S is a measurable set.

Solution:

Let $G = (a, b)$

The symmetric difference

$|G - S| = (G - S) \cup (S - G)$

$\qquad = \emptyset \cup \{a, b\} = \{a, b\}.$

Since $|G - S| = \{a, b\}$ is a finite countable set

$\Rightarrow \ \overline{\mu}(|G - S|) = \overline{\mu}(\{a, b\}) = 0 < \varepsilon , \quad \forall \ \varepsilon \geq 0$

$\therefore$ S is measurable set and

$\mu(S) = \overline{\mu}(S) = b - a$

Theorem (3.4): Properties of Measure

Let $S_1, S_2, \dots , S_n$ be bounded measureable sets, then

(1) $\mu(S) \geq 0$ and $\mu(\emptyset) = 0$.

(2) If $S_1 \subseteq S_2$, then $\mu(S_1) \leq \mu(S_2)$.

(3) $\mu(S_1 \cup S_2) + \mu(S_1 \cap S_2) = \mu(S_1) + \mu(S_2)$.

(4) $\mu(S_1 \cup S_2) \leq \mu(S_1) + \mu(S_2)$.

(5) $\mu(S_1 \cup S_2) = \mu(S_1) + \mu(S_2)$, if S_1 and S_2 are disjoint.

(6) $\mu\left(\bigcup_n S_n\right) \leq \sum_n \mu(S_n)$.

(7) $\mu\left(\bigcup_n S_n\right) = \sum_n \mu(S_n)$, if S_n are disjoint $\forall \ n$.

Example (3.9):

Let $S_1 = \{(-3,-1) \cup (-1,0) \cup (0,2) \cup (3,4) \cup (4,5)\}$ and

$S_2 = \{(-2,-1) \cup (-1,0) \cup (0,1) \cup (3,4)\}$. Satisfy the properties of measure that mention in Theorem (3.4), as much as possible.

Solution:

Since S_1 and S_2 are bounded open sets, then

$$\mu(S_1) = \Delta(S_1) = \sum_{n=1}^{5} \Delta(I_n)$$

$$= \Delta(I_1) + \Delta(I_2) + \Delta(I_3) + \Delta(I_4) + \Delta(I_5)$$

$$= \big(-1-(-3)\big) + \big(0-(-1)\big) + (2-0) + (4-3) + (5-4)$$

$$= 7$$

$$\mu(S_2) = \Delta(S_2) = \sum_{n=1}^{4} \Delta(I_n)$$

$$= \Delta(I_1) + \Delta(I_2) + \Delta(I_3) + \Delta(I_4)$$

$$= (-1-(-2)) + (0-(-1)) + (1-0) + (4-3)$$

$$= 4$$

We have

$S_1 \cup S_2 = \{(-3,-1) \cup (-1,0) \cup (0,2) \cup (3,4) \cup (4,5)\}$

$S_1 \cap S_2 = \{(-2,-1) \cup (-1,0) \cup (0,1) \cup (3,4)\}$

Since $S_1 \cup S_2 = S_1$

$\Rightarrow \mu(S_1 \cup S_2) = \mu(S_1) = 7$

and $S_1 \cap S_2 = S_2$

$\Rightarrow \mu(S_1 \cap S_2) = \mu(S_2) = 4$