Example 2. *Find the Fourier sine transform and Fourier cosine transform of the function $2e^{-5x} + 5e^{-2x}$.*

Sol. Let $f(x) = 2e^{-5x} + 5e^{-2x}$.

$$\therefore \quad \bar{f}_S(s) = \int_0^\infty f(x)\sin sx\,dx = \int_0^\infty (2e^{-5x} + 5e^{-2x})\sin sx\,dx$$

$$= 2\int_0^\infty e^{-5x}\sin sx\,dx + 5\int_0^\infty e^{-2x}\sin sx\,dx$$

$$= 2\,.\,\frac{s}{(5)^2 + s^2} + 5\,.\,\frac{s}{(2)^2 + s^2} \qquad \left(\text{Using } \int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2 + b^2}\right)$$

$$= \mathbf{\frac{2s}{25 + s^2} + \frac{5s}{4 + s^2}}.$$

$$\bar{f}_C(s) = \int_0^\infty f(x)\cos sx\,dx = \int_0^\infty (2e^{-5x} + 5e^{-2x})\cos sx\,dx$$

$$= 2\int_0^\infty e^{-5x}\cos sx\,dx + 5\int_0^\infty e^{-2x}\cos sx\,dx$$

$$= 2\,.\,\frac{5}{(5)^2 + s^2} + 5\,.\,\frac{2}{(2)^2 + s^2} \qquad \left(\text{Using } \int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2 + b^2}\right)$$

$$= \mathbf{10\left(\frac{1}{25 + s^2} + \frac{1}{4 + s^2}\right)}.$$

Example 3. *Find the Fourier sine transform and Fourier cosine transform of the function*

$$f(x) = \begin{cases} \sin x\,, & \text{if } 0 < x < a \\ 0\,, & \text{if } \quad x > a. \end{cases}$$

Sol. Fourier sine transform of $f(x)$

$$= \int_0^\infty f(x)\sin sx\,dx = \int_0^a f(x)\sin sx\,dx + \int_a^\infty f(x)\sin sx\,dx$$

$$= \int_0^a \sin x\sin sx\,dx + \int_a^\infty 0\,.\sin sx\,dx$$

$$= \frac{1}{2}\int_0^a 2\sin sx\sin x\,dx + 0 = \frac{1}{2}\int_0^a (\cos(s-1)x - \cos(s+1)x\,dx$$

$$= \frac{1}{2}\left[\frac{\sin(s-1)x}{s-1} - \frac{\sin(s+1)x}{s+1}\right]\Bigg|_0^a$$

$$= \mathbf{\frac{1}{2}\left[\frac{\sin(s-1)a}{s-1} - \frac{\sin(s+1)a}{s+1}\right]}.$$

Fourier cosine transform of $f(x)$

$$= \int_0^\infty f(x)\cos sx\,dx$$

$$= \int_0^a f(x)\cos sx\,dx + \int_a^\infty f(x)\cos sx\,dx$$

$$= \int_0^a \sin x \cos sx \, dx + \int_a^\infty 0 \,.\, \cos sx \, dx$$

$$= \frac{1}{2}\int_0^a 2\cos sx \sin x \, dx \; + 0 = \frac{1}{2}\int_0^a (\sin (s+1)x - \sin (s-1)x) \, dx$$

$$= \frac{1}{2}\left[-\frac{\cos (s+1)x}{s+1} + \frac{\cos (s-1)x}{s-1}\right]\Bigg|_0^a$$

$$= \frac{1}{2}\left[-\frac{\cos (s+1)a}{s+1} + \frac{\cos (s-1)a}{s-1} + \frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$= \mathbf{\frac{1}{2}\left[\frac{\cos (s-1)a}{s-1} - \frac{\cos (s+1)a}{s+1}\right] - \frac{1}{s^2-1}}.$$

Example 4. *Find the Fourier sine and cosine transforms of the function x^{m-1} , $m > 0$.*

Sol. Let $\quad f(x) = x^{m-1}, m > 0.$

$$\therefore \qquad \bar{f}_S(s) = \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty x^{m-1} \sin sx \, dx \qquad ...(1)$$

and
$$\bar{f}_C(s) = \int_0^\infty f(x) \cos sx \, dx = \int_0^\infty x^{m-1} \cos sx \, dx \qquad ....(2)$$

Let $\qquad g(s) = \int_0^\infty x^{m-1} e^{-isx} \, dx.$

$$y = isx \quad \Rightarrow \quad dy = is\,dx$$

$$\therefore \qquad g(s) = \int_0^\infty \left(\frac{y}{is}\right)^{m-1} e^{-y} \frac{dy}{is} = \frac{1}{(is)^m}\int_0^\infty e^{-y} y^{m-1} \, dy = \frac{1}{(is)^m} \, \Gamma(m)$$

$$= \frac{1}{s^m (e^{i\pi/2})^m} \, \Gamma(m) = \frac{\Gamma(m)}{s^m} \, e^{-im\pi/2} \qquad \left(\because \; e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i\right)$$

$$\therefore \qquad \int_0^\infty x^{m-1} e^{-isx} \, dx = \frac{\Gamma(m)}{s^m} \, e^{-im\pi/2}$$

$$\Rightarrow \qquad \int_0^\infty x^{m-1} (\cos (-sx) + i \sin (-sx) \, dx = \frac{\Gamma(m)}{s^m}\left(\cos\left(\frac{-m\pi}{2}\right) + i \sin\left(\frac{-m\pi}{2}\right)\right)$$

$$\Rightarrow \qquad \int_0^\infty x^{m-1} \cos sx \, dx - i\int_0^\infty x^{m-1} \sin sx \, dx = \frac{\Gamma(m)}{s^m} \cos\frac{m\pi}{2} - i\,\frac{\Gamma(m)}{s^m} \sin\frac{m\pi}{2}$$

Comparing real and imaginary parts, we get

$$\int_0^\infty x^{m-1} \cos sx \, dx = \frac{\Gamma(m)}{s^m} \cos\frac{m\pi}{2} \quad \text{and} \quad \int_0^\infty x^{m-1} \sin sx \, dx = \frac{\Gamma(m)}{s^m} \sin\frac{m\pi}{2}.$$

$$\therefore \quad (1) \quad \Rightarrow \qquad \bar{f}_S(s) = \mathbf{\frac{\Gamma(m)}{s^m} \sin \frac{m\pi}{2}} \quad \text{and} \quad (2) \quad \Rightarrow \quad \bar{f}_C(s) = \mathbf{\frac{\Gamma(m)}{s^m} \cos \frac{m\pi}{2}}.$$

Example 5. *Find the Fourier cosine transform of the function* e^{-x^2}.

Sol. $$F_C\left(e^{-x^2}\right) = \int_0^\infty e^{-x^2} \cos sx\, dx = \text{I, say}$$

$$\therefore \qquad \text{I} = \int_0^\infty e^{-x^2} \cos sx\, dx \qquad ...(1)$$

Differentiating w.r.t. s, using Leibnitz's rule of differentiation under integral sign, we have

$$\frac{d\text{I}}{ds} = \int_0^\infty e^{-x^2} (-x \sin sx)\, dx$$

$$\therefore \qquad \frac{d\text{I}}{ds} = \frac{1}{2}\int_0^\infty \underset{\text{I}}{(\sin sx)}\underset{\text{II}}{(-2x\, e^{-x^2})}\, dx$$

$$= \frac{1}{2}\left[\sin sx\,.\, e^{-x^2} - \int s \cos sx\,.\, e^{-x^2}\, dx\right]_0^\infty$$

$$= \frac{1}{2}(0-0) - \frac{s}{2}\int_0^\infty e^{-x^2} \cos sx\, dx = -\frac{s}{2}\text{I}$$

$$\Rightarrow \qquad \frac{d\text{I}}{ds} + \frac{s}{2}\text{I} = 0 \quad \Rightarrow \quad \frac{d\text{I}}{\text{I}} = -\frac{s}{2}ds \quad \Rightarrow \quad \log \text{I} = -\frac{s^2}{4} + \log c.$$

$$\Rightarrow \qquad \log \frac{\text{I}}{c} = -\frac{s^2}{4} \quad \Rightarrow \quad \frac{\text{I}}{c} = e^{-s^2/4}$$

$$\therefore \qquad \text{I} = ce^{-s^2/4} \qquad ...(2)$$

Putting $s = 0$ in (1) and (2), we get $\int_0^\infty e^{-x^2} \cos 0\, dx = ce^0$

$$\Rightarrow \qquad \int_0^\infty e^{-x^2}\, dx = c \quad \Rightarrow \quad \frac{\sqrt{\pi}}{2} = c \qquad \left(\because \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}\right)$$

$$\therefore \quad (2) \quad \Rightarrow \qquad \text{I} = \frac{\sqrt{\pi}}{2} e^{-s^2/4} \quad \textit{i.e.,} \quad F_C(e^{-x^2}) = \boldsymbol{\frac{\sqrt{\pi}}{2} e^{-s^2/4}}.$$

Example 6. *Find the Fourier cosine transform of the function* $\frac{1}{1+x^2}$. *Hence deduce the Fourier sine transform of the function* $\frac{x}{1+x^2}$.

Sol. $$F_C\left(\frac{1}{1+x^2}\right) = \int_0^\infty \frac{1}{1+x^2} \cos sx\, dx = \text{I, say}$$

$$\therefore \qquad \text{I} = \int_0^\infty \frac{\cos sx}{1+x^2}\, dx \qquad ...(1)$$

Differentiating w.r.t. s, using the Leibnitz's rule of differentiation under integral sign, we have

$$\frac{dI}{ds} = \int_0^\infty \frac{-x \sin sx}{1+x^2}\, dx \qquad \text{...(2)}$$

We have $$-\frac{x}{1+x^2} = -\frac{x^2}{x(1+x^2)} = -\frac{(1+x^2)-1}{x(1+x^2)} = -\frac{1}{x} + \frac{1}{x\,(1+x^2)}$$

$\therefore$ (2) $\Rightarrow$ $$\frac{dI}{ds} = \int_0^\infty \left(-\frac{1}{x} + \frac{1}{x(1+x^2)}\right) \sin sx\, dx$$

$$= -\int_0^\infty \frac{\sin sx}{x}\, dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\, dx$$

$\therefore$ $$\frac{dI}{ds} = -\frac{\pi^*}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}\, dx \qquad \text{...(3)}$$

Differentiating again w.r.t. s, we get

$$\frac{d^2I}{ds^2} = 0 + \int_0^\infty \frac{x \cos sx}{x(1+x^2)}\, dx = \int_0^\infty \frac{\cos sx}{1+x^2}\, dx = I$$

$\Rightarrow$ $$(D^2 - 1)I = 0, \quad \text{where} \quad D \equiv \frac{d}{ds}$$

$$D^2 - 1 = 0 \quad \Rightarrow \quad D = \pm 1$$

$\therefore$ $$I = c_1\, e^{1.s} + c_2\, e^{-1.s} = c_1 e^s + c_2\, e^{-s}$$

$\Rightarrow$ $$\frac{dI}{ds} = c_1 e^s - c_2 e^{-s}$$

Putting $s = 0$ in (1), we get

$$c_1 e^0 + c_2 e^0 = \int_0^\infty \frac{\cos 0}{1+x^2}\, dx = \int_0^\infty \frac{1}{1+x^2}\, dx = \tan^{-1} x \Big|_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$\therefore$ $$c_1 + c_2 = \frac{\pi}{2} \qquad \text{...(4)}$$

Putting $s = 0$ in (3), we get

$$c_1 e^0 - c_2 e^0 = -\frac{\pi}{2} + \int_0^\infty \frac{\sin 0}{x(1+x^2)}\, dx = -\frac{\pi}{2} + \int_0^\infty 0\, dx = -\frac{\pi}{2}$$

$\therefore$ $$c_1 - c_2 = -\frac{\pi}{2} \qquad \text{...(5)}$$

Solving (4) and (5), we get $c_1 = 0$ and $c_2 = \dfrac{\pi}{2}$.

$\therefore$ $$I = 0.\, e^s + \frac{\pi}{2}\, e^{-s} = \frac{\pi}{2} e^{-s}$$

***Why this step.** We know that $\displaystyle\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}$.

Let $y = sx$, $s > 0$.

$\therefore$ $$\int_0^\infty \frac{\sin sx}{x}\, dx = \int_0^\infty \frac{\sin y}{y/s} \frac{dy}{s} = \int_0^\infty \frac{\sin y}{y}\, dy = \frac{\pi}{2}.$$

$$\therefore \qquad F_C\left(\frac{1}{1+x^2}\right) = \frac{\pi}{2}\mathbf{e^{-s}}.$$

Also $$F_S\left(\frac{x}{1+x^2}\right) = \int_0^\infty \frac{x}{1+x^2}\sin sx\,dx = -\frac{dI}{ds} \qquad \text{(Using (2))}$$

$$= -(c_1e^s - c_2e^{-s}) = -\left(0.e^s - \frac{\pi}{2}e^{-s}\right) = \frac{\pi}{2}\mathbf{e^{-s}}.$$

Example 7. *Find the Fourier sine transform of the function* $\dfrac{1}{x(x^2+a^2)}$.

Sol. $$F_S\left(\frac{1}{x(x^2+a^2)}\right) = \int_0^\infty \frac{1}{x(x^2+a^2)}\sin sx\,dx = \text{I, say}$$

$$\therefore \qquad I = \int_0^\infty \frac{\sin sx}{x(x^2+a^2)}\,dx \qquad ...(1)$$

Differentiating w.r.t. s, using Leibnitz's rule of differentiation under integral sign, we have

$$\frac{dI}{ds} = \int_0^\infty \frac{x\cos sx}{x(x^2+a^2)}\,dx = \int_0^\infty \frac{\cos sx}{x^2+a^2}\,dx \qquad ...(2)$$

Differentiating again w.r.t. s, we get

$$\frac{d^2I}{ds^2} = \int_0^\infty \frac{-x\sin sx}{x^2+a^2}\,dx$$

We have $$-\frac{x}{x^2+a^2} = -\frac{x^2}{x(x^2+a^2)} = -\frac{(x^2+a^2)-a^2}{x(x^2+a^2)} = -\frac{1}{x} + \frac{a^2}{x(x^2+a^2)}$$

$$\therefore \qquad \frac{d^2I}{ds^2} = \int_0^\infty \left(-\frac{1}{x} + \frac{a^2}{x(x^2+a^2)}\right)\sin sx\,dx$$

$$= -\int_0^\infty \frac{\sin sx}{x}\,dx + a^2\int_0^\infty \frac{\sin sx}{x(x^2+a^2)}\,dx = -\frac{\pi}{2} + a^2 I$$

$$\therefore \qquad \frac{d^2I}{ds^2} - a^2 I = -\frac{\pi}{2} \qquad \left(\text{Using } \int_0^\infty \frac{\sin \lambda x}{x}\,dx = \frac{\pi}{2}, \text{ if } \lambda > 0\right)$$

$$\Rightarrow \qquad (D^2 - a^2)I = -\frac{\pi}{2}$$

$$D^2 - a^2 = 0 \quad \Rightarrow \quad D = \pm a \quad \therefore \quad \text{C.F.} = c_1e^{as} + c_2\,e^{-as}$$

$$\text{P.I.} = \frac{1}{D^2-a^2}\left(-\frac{\pi}{2}\right) = -\frac{\pi}{2}\cdot\frac{1}{D^2-a^2}e^{0s} = -\frac{\pi}{2}\cdot\frac{1}{0^2-a^2}e^{0s} = \frac{\pi}{2a^2}$$

$$\therefore \qquad I = \text{C.F.} + \text{P.I.} = c_1e^{as} + c_2e^{-as} + \frac{\pi}{2a^2}$$

$$\Rightarrow \qquad \frac{dI}{ds} = ac_1e^{as} - ac_2e^{-as}$$

Putting $s = 0$ in (1), we get

$$\int_0^\infty \frac{\sin 0}{x(x^2+a^2)}\,dx = c_1e^0 + c_2e^0 + \frac{\pi}{2a^2}.$$

$$\Rightarrow \qquad 0 = c_1 + c_2 + \frac{\pi}{2a^2} \quad \Rightarrow \quad c_1 + c_2 = -\frac{\pi}{2a^2} \qquad \ldots(3)$$

Putting $s = 0$ in (2), we get

$$\int_0^\infty \frac{\cos 0}{x^2 + a^2}\, dx = ac_1 e^0 - ac_2 e^0.$$

$$\Rightarrow \qquad \frac{1}{a} \tan^{-1} \frac{x}{a} \Big|_0^\infty = ac_1 - ac_2 \quad \Rightarrow \quad \frac{1}{a}\left(\frac{\pi}{2} - 0\right) = a(c_1 - c_2)$$

$$\Rightarrow \qquad c_1 - c_2 = \frac{\pi}{2a^2} \qquad \ldots(4)$$

Solving (3) and (4), we get $c_1 = 0$ and $c_2 = -\dfrac{\pi}{2a^2}$.

$$\therefore \qquad I = 0.e^{as} + \left(-\frac{\pi}{2a^2}\right) e^{-as} + \frac{\pi}{2a^2} = \frac{\pi}{2a^2}(1 - e^{-as})$$

$$\therefore \qquad F_S\left(\frac{1}{x(x^2 + a^2)}\right) = \boldsymbol{\frac{\pi}{2a^2}(1 - e^{-as})}.$$

TEST YOUR KNOWLEDGE

Find the Fourier sine and cosine transforms of the following functions (Q. 1– 6) :

1. $f(x) = e^{-7x}$

2. $f(x) = 3e^{-2x} + 8e^{-11x}$

3. $f(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ 0, & \text{if } \quad x > 1 \end{cases}$

4. $f(x) = \begin{cases} 0, & \text{if } 0 < x < a \\ x, & \text{if } a \le x \le b \\ 0, & \text{if } \quad x > b \end{cases}$

5. $f(x) = \begin{cases} x, & \text{if } 0 < x < 1 \\ 2 - x, & \text{if } 1 < x < 2 \\ 0, & \text{if } \quad x > 2 \end{cases}$

6. $f(x) = \begin{cases} \cos x, & \text{if } 0 < x < a \\ 0, & \text{if } \quad x > a \end{cases}$.

7. Find the Fourier sine transform of the function $\dfrac{1}{x}$.

8. Find the Fourier sine transform of the function $\dfrac{e^{-ax}}{x}$, $a > 0$.

9. If $\bar{f}_S(s)$ and $\bar{f}_C(s)$ be the Fourier sine and cosine transforms of the function $f(x)$ respectively, then show that :

(*i*) $F_S(f(x) \sin ax) = \frac{1}{2}[\bar{f}_C(s-a) - \bar{f}_C(s+a)]$ (*ii*) $F_S(f(x) \cos ax) = \frac{1}{2}[\bar{f}_S(s+a) + \bar{f}_S(s-a)]$

(*iii*) $F_C(f(x) \sin ax) = \frac{1}{2}[\bar{f}_S(s+a) - \bar{f}_S(s-a)]$ (*iv*) $F_C(f(x) \cos ax) = \frac{1}{2}[\bar{f}_C(s+a) + \bar{f}_C(s-a)]$.

Answers

1. $\bar{f}_S(s) = \dfrac{s}{49 + s^2}$, $\bar{f}_C(s) = \dfrac{7}{49 + s^2}$

2. $\bar{f}_S(s) = s\left(\dfrac{3}{4+s^2} + \dfrac{8}{121+s^2}\right)$, $\bar{f}_C(s) = 2\left(\dfrac{3}{4+s^2} + \dfrac{44}{121+s^2}\right)$

3. $\bar{f}_S(s) = \dfrac{1-\cos s}{s}$, $\bar{f}_C(s) = \dfrac{\sin s}{s}$

4. $\bar{f}_S(s) = \dfrac{1}{s}(a \cos as - b \cos bs) + \dfrac{1}{s^2}(\sin bs - \sin as)$,

$\bar{f}_C(s) = \dfrac{1}{s}(b \sin bs - a \sin as) + \dfrac{1}{s^2}(\cos bs - \cos as)$

5. $\bar{f}_S(s) = \dfrac{2 \sin s(1-\cos s)}{s^2}$, $\bar{f}_C(s) = \dfrac{1}{s^2}(2 \cos s - \cos 2s - 1)$

6. $\bar{f}_S(s) = \dfrac{s}{s^2-1} - \dfrac{1}{2}\left[\dfrac{\cos(s+1)a}{s+1} + \dfrac{\cos(s-1)a}{s-1}\right]$, $\bar{f}_C(s) = \dfrac{1}{2}\left[\dfrac{\sin(s+1)a}{s+1} + \dfrac{\sin(s-1)a}{s-1}\right]$

7. $\dfrac{\pi}{2}$ **8.** $\tan^{-1}\dfrac{s}{a}$.

Hint

9. (*i*) $F_S(f(x) \sin ax)$
$$= \int_0^\infty (f(x) \sin ax) \sin sx \, dx$$
$$= \frac{1}{2}\int_0^\infty f(x)\,(2 \sin sx \sin ax)\, dx$$
$$= \frac{1}{2}\int_0^\infty f(x)\,(\cos(s-a)x - \cos(s+a)x)\, dx$$
$$= \frac{1}{2}\left[\int_0^\infty f(x) \cos(s-a)x\, dx - \int_0^\infty f(x)\cos(s+a)x\, dx\right]$$
$$= \frac{1}{2}\left[\bar{f}_C(s-a) - \bar{f}_C(s+a)\right].$$

TYPE II. Problems Based on Inverse Fourier Sine and Cosine Transforms

ILLUSTRATIVE EXAMPLES

Example 1. *Find the function whose Fourier cosine transform is* $\dfrac{\sin as}{s}$, $a > 0$.

Sol. Let $\dfrac{\sin as}{s}$ be the Fourier cosine transform of $f(x)$, $x > 0$.

$$\therefore \quad f(x) = F_C^{-1}\left(\frac{\sin as}{s}\right) = \frac{2}{\pi}\int_0^\infty \frac{\sin as}{s}\cos sx\, ds$$
$$= \frac{1}{\pi}\int_0^\infty \frac{2 \sin as \cos sx}{s}\, ds = \frac{1}{\pi}\int_0^\infty \frac{\sin s(a+x) + \sin s(a-x)}{s}\, ds$$
$$= \frac{1}{\pi}\int_0^\infty \frac{\sin s(a+x)}{s}\, ds + \frac{1}{\pi}\int_0^\infty \frac{\sin s(a-x)}{s}\, ds$$

$$= \begin{cases} \dfrac{1}{\pi}\cdot\dfrac{\pi}{2}+\dfrac{1}{\pi}\cdot\dfrac{\pi}{2} & \text{if } a-x>0 \\ \dfrac{1}{\pi}\cdot\dfrac{\pi}{2}+\dfrac{1}{\pi}\cdot\left(-\dfrac{\pi}{2}\right) & \text{if } a-x<0 \end{cases} \qquad (\because \quad x>0, a>0 \ \Rightarrow \ x+a>0)$$

$$= \begin{cases} 1, & \text{if } x<a \\ 0, & \text{if } x>a \end{cases}$$

$$\therefore \qquad f(x) = \begin{cases} \mathbf{1}, & \textbf{if } \mathbf{0<x<a} \\ \mathbf{0}, & \textbf{if } \quad \mathbf{x>a} \end{cases}. \qquad (\because \quad x>0)$$

Remark. If $\lambda < 0$, then

$$\int_0^\infty \frac{\sin \lambda x}{x}\,dx = \int_0^\infty \frac{-\sin(-\lambda)x}{x}\,dx = -\int_0^\infty \frac{\sin(-\lambda)x}{x}\,dx = -\frac{\pi}{2}.$$

Example 2. *Find the function f(x) if* $F_C(f(x)) = \begin{cases} \dfrac{1}{2\pi}\left(a-\dfrac{s}{2}\right), & \text{if } 0<s<2a \\ 0, & \text{if } \quad s \ge 2a \end{cases}.$

Sol. We have $\quad F_C(f(x)) = \begin{cases} \dfrac{1}{2\pi}\left(a-\dfrac{s}{2}\right), & \text{if } 0<s<2a \\ 0, & \text{if } \quad s \ge 2a \end{cases}.$

$$\therefore \qquad f(x) = F_C^{-1}(F_C(f(x))) = \frac{2}{\pi}\int_0^\infty F_C(f(x))\cos sx\,ds$$

$$= \frac{2}{\pi}\left[\int_0^{2a} F_C(f(x))\cos sx\,ds + \int_{2a}^\infty F_C(f(x))\cos sx\,ds\right]$$

$$= \frac{2}{\pi}\left[\int_0^{2a} \frac{1}{2\pi}\left(a-\frac{s}{2}\right)\cos sx\,ds + \int_{2a}^\infty 0\,.\cos sx\,ds\right]$$

$$= \frac{1}{\pi^2}\int_0^{2a}\left(a-\frac{s}{2}\right)\cos sx\,ds + 0$$

$$= \frac{1}{\pi^2}\left[\left(a-\frac{s}{2}\right)\frac{\sin sx}{x} - \int\left(-\frac{1}{2}\right)\frac{\sin sx}{x}\,ds\right]\Bigg|_0^{2a}$$

$$= \frac{1}{\pi^2}\left[\frac{2a-s}{2x}\sin sx - \frac{\cos sx}{2x^2}\right]_0^{2a} = \frac{1}{\pi^2}\left[0-\frac{\cos 2ax}{2x^2}-0+\frac{\cos 0}{2x^2}\right]$$

$$= \frac{1}{2\pi^2x^2}(1-\cos 2ax) = \frac{\mathbf{\sin^2 ax}}{\mathbf{\pi^2 x^2}}.$$