Masses and Moments in Three Dimensions
الكتل والعزوم في الابعاد الثلاثية

Example 1: Find the center of mass of a solid of constant density δ bounded below by the disk (القرص) $x^2 + y^2 \leq 4$ in the plane $z = 0$ and above by the paraboloid (الجسم المكافئ الدوراني) $z = 4 - x^2 - y^2$.

Solution:

$$M = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{4-x^2-y^2} \delta \, dz \, dy \, dx = \iint_R \int_0^{4-x^2-y^2} \delta \, dz \, dy \, dx$$

$$\iint_R \delta \;\; Z|_0^{4-x^2-y^2} \;\; dy \, dx = \delta \iint_{\boldsymbol{R}} \; [4 - x^2 - y^2] \; dy \, dx$$

Polar Coordinates

$$\delta \iint_{\boldsymbol{R}} \; [4 - x^2 - y^2] \; dy \, dx = \delta \int_0^{2\pi} \int_0^{2} (4 - r^2) \; r \, dr \, d\theta$$

$$= \delta \int_0^{2\pi} \int_0^{2} (4r - r^3) \; dr \, d\theta = \delta \int_0^{2\pi} \left. \frac{4r^2}{2} - \frac{r^4}{4} \right|_0^2 \; d\theta = \delta \int_0^{2\pi} 8 - 4 \, d\theta$$

$$= \delta \int_0^{2\pi} 4 \, d\theta = \;\; \delta \;\; 4\theta|_0^{2\pi} = \;\; 8\pi\delta$$

The mass is $M = \; 8\pi\delta$

$$M_{\boldsymbol{yz}} = \iint_R \int_0^{4-x^2-y^2} \delta \, \boldsymbol{x} \, dz \, dy \, dx = \delta \iint_R \;\; xz|_0^{4-x^2-y^2} \;\; dy \, dx$$

$$\delta \iint_{\boldsymbol{R}} \; x[4 - x^2 - y^2] \; dy \, dx =$$

$$\delta \int_0^{2} \int_0^{2\pi} (r \cos\theta)(4 - r^2) \, r \, d\theta dr = \delta \int_0^{2} \int_0^{2\pi} (4r^2 \cos\theta - r^4 \cos\theta) d\theta \, dr$$

$$= \delta \int_0^2 4r^2 sin\theta - r^4 sin\theta|_0^{2\pi}\ d\theta dr = \delta \int_0^2 [0]\ dr = 0$$

$$\therefore \quad M_{yz} = 0$$

$$M_{xz} = \iint\limits_R \int_0^{4-x^2-y^2} \delta\, \boldsymbol{y}\, dz\, dy\, dx = \delta \iint_R \boldsymbol{y}z|_0^{4-x^2-y^2}\ dy\, dx$$

$$= \delta \iint_{\boldsymbol{R}} \boldsymbol{y}[4 - x^2 - y^2]\ dy\, dx =$$

$$\delta \int_0^2 \int_0^{2\pi} (r \sin\theta)(4 - r^2)\, r\, d\theta dr = \delta \int_0^2 \int_0^{2\pi} (4r^2 \sin\theta - r^4 \sin\theta) d\theta\, dr$$

$$= \delta \int_0^2 -4r^2 \cos\theta + r^4 \cos\theta|_0^{2\pi}\ dr$$

$$\delta \int_0^2 \left[-4r^2 + r^4 - [-4r^2 + r^4]\right]\ dr = \delta \int_0^2 [-4r^2 + r^4 + 4r^2 - r^4]\, dr$$

$$= \delta \int_0^2 [0]\, dr = 0$$

$$\therefore \quad M_{xz} = 0$$

$$M_{xy} = \iint\limits_R \int_0^{4-x^2-y^2} \delta\, \boldsymbol{z}\, dz\, dy\, dx = \delta \iint_R \left.\frac{z^2}{2}\right|_0^{4-x^2-y^2}\ dy\, dx$$

$$\frac{\delta}{2} \iint_{\boldsymbol{R}} [4 - x^2 - y^2]^2\ dy\, dx = \frac{\delta}{2} \int_0^{2\pi} \int_0^2 (4 - r^2)^2\ r\, dr\, d\theta$$

$$= \frac{\delta}{2(-\mathbf{2})} \int_0^{2\pi} \int_0^2 (-\mathbf{2})(4 - r^2)^2\ r\, dr\, d\theta = \frac{\delta}{-4} \int_0^{2\pi} \left.\frac{(4 - r^2)^3}{3}\right|_0^2 d\theta$$

$$\frac{\delta}{-12} \int_0^{2\pi} [(4-4)^3 - (4-0)^3]\ d\theta = \frac{\delta}{-12} \int_0^{2\pi} -64\ d\theta = \frac{64\delta}{12} \int_0^{2\pi} d\theta$$

$$= \frac{32\,\delta}{6} \int_0^{2\pi} d\theta \quad = \frac{32\,\delta}{6} \theta|_0^{2\pi} \quad = \frac{32\,\delta}{6}\ \ 2\pi \quad = \frac{32\pi\,\delta}{3}$$

$$\therefore \quad M_{xy} = \frac{32\pi\,\delta}{3}$$

Center of mass is

$$\bar{X} = \frac{M_{yz}}{M} = \frac{0}{8\pi\delta} = 0 \ , \qquad \bar{y} = \frac{M_{xz}}{M} = \frac{0}{8\pi\delta} = 0 \ ,$$

$$\bar{\boldsymbol{Z}} = \frac{M_{xy}}{M} = \frac{\frac{32\pi\ \delta}{3}}{8\pi\delta} = \frac{4}{3}$$

Example 2: Find the mass in the first octant of the solid inside the cylinder $x^2 + y^2 = a^2$ and between the planes $z = 0, z = h$, and has density $\delta = x$.

Solution:

$$M = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^h \delta \ dz\ dy\ dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^h x\ dz\ dy\ dx$$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} xz|_0^h \ \ dy\ dx = \int_0^a \int_0^{\sqrt{a^2-x^2}} xh\ \ dy\ dx = \int_{\boldsymbol{0}}^{\boldsymbol{a}} xh\ \ y|_0^{\sqrt{a^2-x^2}}\ dx$$

$$= \int_{\boldsymbol{0}}^{\boldsymbol{a}} xh(a^2 - x^2)^{\frac{1}{2}}\ dx = \frac{h}{(-\boldsymbol{2})} \int_{\boldsymbol{0}}^{\boldsymbol{a}} (-\boldsymbol{2})x\ \ (a^2 - x^2)^{\frac{1}{2}} = \frac{h}{(-\boldsymbol{2})}\ \frac{(a^2 - x^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^a$$

$$\frac{-h}{3}\left[(a^2 - a^2)^{\frac{3}{2}} - (a^2 - 0)^{\frac{3}{2}}\right] = \frac{-h}{3}(-a^3\ ) = \frac{a^3 h}{3}$$

$$\therefore \quad M = \frac{a^3 h}{3}$$