## ***13th Lecture***

**Example (4.7):** If $E = [a, b]$ is a subset in $(R, \tau)$. Then $E$ is countably compact.

**Solution:** Let $A \subset E$ be infinite

$\Rightarrow\ A \subset [a, b]$

Since $E$ is closed and bounded

$\Rightarrow A$ is bounded

$\Rightarrow\ sup\ A$ exists point $x = sup\ A$

(Every bounded subset of $R$ has a limit point in $R$)

$\Rightarrow\ A$ has a limit point $x = sup\ A$

$A \subset E\ \Rightarrow\ d(A) \subset d(E) = E$

$\Rightarrow\ d(A) \subset E\ \Rightarrow\ x \in E$

$\Rightarrow\ \forall^{infinite}$ subset $A$ of $E$ has a limit point $x \in E$

$\Rightarrow\ E$ is countably compact

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**Theorem (4.6):** Every compact space is countably compact.

**Proof:** Let $(X, \tau)$ be a compact space

Let $E \subset X$ be any infinite subset of $X$

Assume that $E$ has no limit point in $X$

$\Rightarrow\ \forall\ x \in X, x \notin d(E)$

$\Rightarrow\ \forall\ x \in X, \exists^{open}\ G_x \ni x; (G_x \cap E) \backslash \{x\} = \emptyset$

$\Rightarrow\ \forall\ x \in X, \exists^{open}\ G_x \ni x; (G_x \cap E) = \{x\}$

$\Rightarrow\ G_x \cap E$ contains are point (at most) $x$

$\Rightarrow\ \{G_x\}_{x \in X}$ is an open cover for $X$ (because $X = \bigcup_{x \in X} G_x$)

$\Rightarrow\ X = \bigcup_{i=1}^{n} G_{x_i}$

$\Rightarrow\ E \cap X = E \cap (\bigcup_{i=1}^{n} G_{x_i})$

$\Rightarrow\ E = \bigcup_{i=1}^{n}(E \cap G_{x_i})$

$\Rightarrow\ E = \bigcup_{i=1}^{n}(\{x_i\})$

$\Rightarrow\ E = \{x_1, x_2, x_3, \dots, x_n\}$

$\Rightarrow\ E$ is finite

$\Rightarrow$ Contradiction to our assumption

$\Rightarrow\ E$ has a limit point in $X$

$\Rightarrow\ (X, \tau)$ countably compact

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**Theorem (4.7):** Every sequentially compact space is countably compact.

**Proof:**

Let $(X, \tau)$ be a sequentially compact topological space and

Let $\emptyset \neq A \subset X$ be infinite subset of $X$

$\Rightarrow\ \exists\ \{a_n\} = \{a_1, a_2, \dots\}$ in $A$

Since $X$ is sequentially compact

$\Rightarrow \exists$ subsequence $\{a_{n_k}\}$ of $\{a_n\}$ such that

$\{a_{n_k}\}$ converges to element $p \in X$

Now, by definition of convergent sequence, every open set contain $p$ contains infinite number of elements of $\{a_{n_k}\}$

$\forall^{open}\ G \ni p$ (because the element of sequence is different)

Then

Every open set contain $p$ contains at the infinite number of element of $A$

Thus $p \in X$ is a limit point of $A\ \Rightarrow A$ is countably compact.

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**Definition (4.6): (Locally Compactness)** **التراص المحلي**

We say that a topological space $(X, \tau)$ is **locally compact** iff every point $x \in X$ is contained in a compact neighborhood (nbhd).

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**Examples (4.8):** The usual topological space $(R, \tau)$ is locally compact. Because $\forall\ x \in R$ we can find a compact nbhd say $[x - \varepsilon, x + \varepsilon] \ni X$.

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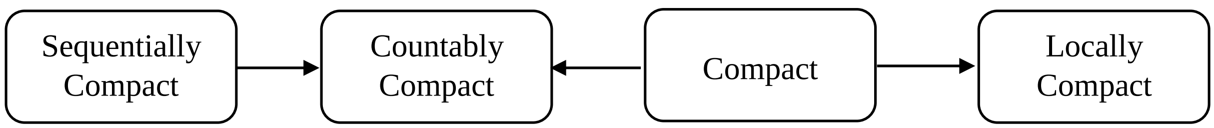
**Note (4.1):**

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**Exercises (4.1): (Homework)**

(1) Prove that $E = [0, \infty)$ is compact in $(R, \tau)$.

(2) If $X \neq \emptyset$ and $\tau = \{\emptyset, G \subset X : G^c\ is\ finite\}$ prove or disprove that $(X, \tau)$ is compact.

(3) Prove or disprove that a subset of a compact set is compact.

(4) Every finite subset of $(X, \tau)$ is sequentially compact.

(5) Every compact space is locally compact.

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