Corollary (5.1): Let $f:[a,b] \to R$ is continuous function, then f is uniformly continuous.

Example (5.11): Show that the function $f(x) = x^2$ is uniformly continuous in $[-M, M]$.

Solution: Let $\varepsilon > 0$, $\exists\, \delta > 0$, s.t.

$d(x, x_0) = |x - x_0| < \delta$

$$\Rightarrow\ d\big(f(x), f(x_0)\big) = |f(x) - f(x_0)|$$
$$= |x^2 - x_0^2|$$
$$= |(x + x_0)(x - x_0)|$$
$$= |x + x_0||x - x_0|$$
$$\leq \delta(|x| + |x_0|)$$
$$\leq 2M\delta$$

Since $\varepsilon = 2M\delta$ $\Rightarrow$ f is uniformly continuous function.

Theorem (5.9): Every uniformly continuous function is continuous.

Proof:

Let $f: X \to Y$ is uniformly continuous then

$\Rightarrow\ \forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $\forall\, x, x_0 \in X$,

if $d_X(x, x_0) < \delta \quad \Rightarrow \quad d_Y\big(f(x), f(x_0)\big) < \varepsilon$.

$\Rightarrow$ f is continuous at each $x_0 \in X$

$\Rightarrow$ f is continuous on X

Remark (5.2): The converse of the above theorem is not true as shown in the following example.

Example (5.12): Let $f:(0,1) \to R$, defined by $f(x) = \frac{1}{x}$, determine wether f is uniformly continuous or not.

Solution:

Let $\varepsilon > 0$, $\exists\ \delta > 0$, s.t.

$d(x, x_0) = |x - x_0| < \delta$, set $x_0 = \frac{x}{2}$

$\Rightarrow\ d\big(f(x), f(x_0)\big) = |f(x) - f(x_0)|$

$$= \left|\frac{1}{x} - \frac{1}{x_0}\right|$$

$$= \left|\frac{1}{x} - \frac{2}{x}\right|$$

$$= \left|\frac{1}{x}\right| \geq 1 \ ,\ \forall\ x \in (0,1)$$

$\Rightarrow\ f$ is continuous but it is not uniformly continuous function.

Chapter Six

متتابعات ومتسلسلات الدوال

Sequence and Series Functions

Definition (6.1): (Sequence of Functions) **متتابعات الدوال**

Let $S \subseteq R$ and $F = \{f : f: S \to R\}$. Let $< f_n >$ be a **sequence of functions** in S. Then $< f_n(x) >$ is a **sequence** in R, $\forall\, x \in S$.

Definition (6.2): (Pointwise Convergence) **التقارب النقطي**

We say that the sequence $< f_n >$ **convergent pointwise** to f if $\forall\, x \in S$ the sequence of numbers $< f_n(x) >$ converges to $f(x)$.

i.e. $\forall\, \varepsilon > 0, \exists\, k \in N, k = k(\varepsilon, x)$ such that $|f_n(x) - f(x)| < \varepsilon\, , \forall\, n > k$

i.e. $\lim_{n\to\infty} f_n = f$ pointwise iff $\lim_{n\to\infty} f_n(x) = f(x), \forall\, x \in X$

Example (6.1): Let $f_n: [0,1] \to R$, defined by $f_n(x) = e^{\frac{x}{n}}$, $\forall\, x \in [0,1]$, $\forall\, n \in N$, show that the sequence of function $< f_n >$ is converge pointwise.

Solution:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} e^{\frac{x}{n}}$$

$$= e^0 = 1 \quad \text{for} \quad x \in [0,1]$$

Thus $f_n(x)$ converges pointwise to $f(x) = 1$

Example (6.2): Let $f_n: [0,1] \to R$, defined by $f_n(x) = x^n$, $\forall\, x \in [0,1]$, $\forall\, n \in N$, show that the sequence of function $< f_n >$ is converge pointwise on [0,1].

Solution:

Since

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} \lim\limits_{n\to\infty} x^n = 0 & if\ x \in [0,1) \\ \lim\limits_{n\to\infty} 1^n = 1 & if\quad x = 1 \end{cases}$$

Thus $f_n(x)$ converges pointwise to $g(x) = \begin{cases} 1 & if\quad x = 1 \\ 0 & if\quad x \in [0,1) \end{cases}$

Example (6.3): Let $f_n: [-1,1] \to R$, defined by $f_n(x) = x^n$, $\forall\, x \in [-1,1]$, $\forall\, n \in N$, determine whether the sequence of functions $< f_n >$ is converge pointwise on $[-1,1]$ or not.

Solution:

Since

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \begin{cases} \lim\limits_{n\to\infty} 1^n = 1 & if\quad x = 1 \\ \lim\limits_{n\to\infty} x^n = 0 & if\ x \in (-1,1) \\ \lim\limits_{n\to\infty} (-1)^n = indetermine & if\quad x = -1 \end{cases}$$

$\Rightarrow$ $f_n(x)$ does not converge pointwise.

Example (6.4): Let $< f_n >$ be a sequence of function on R defined by

$f_n(x) = \frac{x}{n}$, $\forall\, x \in R$, $\forall\, n \in N$

Solution:

Since $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{x}{n} = 0$, $\forall\ x \in R$

Thus $f_n(x)$ converges pointwise to $f(x) = 0$

Example (6.5): Let $< f_n >$ be a sequence of function on R defined by

$f_n(x) = \frac{nx}{1+nx}$, $\forall\, x \in [0,1]$, $\forall\, n \in N$

Solution:

Since $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{nx}{1+nx} = 1$, $\forall\, x \in [0,1]$

Thus $f_n(x)$ converges pointwise to $f(x) = 1$.
