Now,

(1) $\mu(S_1) = 7 > 0$ and $\mu(S_2) = 4 > 0$.

(2) Since $S_2 \subseteq S_1 \Rightarrow \mu(S_2) = 4 < 7 = \mu(S_1)$.

(3) $\mu(S_1 \cup S_2) + \mu(S_1 \cap S_2) = 7 + 4$

$$= \mu(S_1) + \mu(S_2)$$

(4) $\mu(S_1 \cup S_2) = 7 < 11 = 7 + 4 = \mu(S_1) + \mu(S_2)$

Example (3.10):

Let $S_1 = \{(-2,-1) \cup (4,5)\}$, $S_2 = \{(-1,0) \cup (2,3) \cup (3,4)\}$ and $S_3 = \{(-3,-2) \cup (0,2) \cup (5,6)\}$. Show that

$$\mu\left(\bigcup_{n=1}^{3} S_n\right) = \sum_{n=1}^{3} \mu(S_n)$$

Solution:

Since S_1, S_2 and S_3 are bounded open sets, then

$$\mu(S_1) = \Delta(S_1) = \sum_{n=1}^{2} \Delta(I_n)$$
$$= \Delta(I_1) + \Delta(I_2)$$
$$= \big(-1 - (-2)\big) + (5 - 4) = 2$$

$$\mu(S_2) = \Delta(S_2) = \sum_{n=1}^{3} \Delta(I_n)$$
$$= \Delta(I_1) + \Delta(I_2) + \Delta(I_3)$$
$$= (0 - (-1)) + (3 - 2) + (4 - 3) = 3$$

$$\mu(S_3) = \Delta(S_3) = \sum_{n=1}^{3} \Delta(I_n)$$
$$= \Delta(I_1) + \Delta(I_2) + \Delta(I_3)$$

$$= (-2-(-3)) + (2-0) + (6-5) = 4$$

We have

$$\bigcup_{n=1}^{3} S_n = S_1 \cup S_2 \cup S_3$$

$$= \{(-3,-2) \cup (-2,-1) \cup (-1,0) \cup (0,2) \cup (2,3) \cup (3,4) \cup (4,5) \cup (5,6)\}$$

$$\Rightarrow \mu\left(\bigcup_{n=1}^{3} S_n\right) = \mu(S_1 \cup S_2 \cup S_3)$$

$$= \Delta(S_1 \cup S_2 \cup S_3)$$

$$= \sum_{n=1}^{6} \Delta(I_n)$$

$$= \Delta(I_1) + \Delta(I_2) + \Delta(I_3) + \Delta(I_4) + \Delta(I_5) + \Delta(I_6)$$

$$= \big(-2-(-3)\big) + \big(-1-(-2)\big) + \big(0-(-1)\big) + (2-0)$$

$$+(3-2) + (4-3) + (5-4) + (6-5) = 9$$

And $\bigcap_{n=1}^{3} S_n = S_1 \cap S_2 \cap S_3 = \emptyset$

$$\Rightarrow \mu\left(\bigcap_{n=1}^{3} S_n\right) = \mu(S_1 \cap S_2 \cap S_3)$$

$$= \Delta(S_1 \cap S_2 \cap S_3)$$

$$= \Delta(\emptyset) = 0$$

Now,

$$\sum_{n=1}^{3} \mu(S_n) = \mu(S_1) + \mu(S_2) + \mu(S_3)$$

$$= 2 + 3 + 4 = 9$$

$$\therefore \mu\left(\bigcup_{n=1}^{3} S_n\right) = 9 = \sum_{n=1}^{3} \mu(S_n).$$

Theorem (3.5):

(1) All intervals are measurable and the measure of an interval is its length.

(2) All open and closed sets are measurable.

(3) The complement of a measurable set is measurable.

(4) Every set with outer measure zero is measurable, i.e. $\mu(S) = 0$.

(5) The union and intersection of a finite or countable number of measurable sets is measurable.

Notes (3.4):

(1) **Empty** set is measurable and $\mu(\emptyset) = 0$.

(2) **Natural** numbers set is measurable and $\mu(N) = 0$.

(3) **Integer** numbers set is measurable and $\mu(Z) = 0$.

(4) **Rational** numbers set is measurable and $\mu(Q) = 0$.

(5) **Irrational** numbers set is measurable and $\mu(Q') = \infty$.

(6) **Real** numbers set is measurable and $\mu(R) = \infty$.

(7) **Complex** numbers set is measurable and $\mu(\mathbb{C}) = \infty$.

مهملة

Definition (3.7): Let $S \subseteq R$. Then S is called **negligible set** if $\mu(S) = 0$.

Lemma (3.2):

(1) The empty set is negligible set.

(2) Every finite subset of R is negligible set.

(3) Every countable infinite subset of R is negligible set.

(4) Every subset of a negligible set is negligible set.

(5) Any intersection of negligible sets is negligible.

(6) Finite union of negligible sets is negligible set.
