Example 3. *Find the inverse Fourier sine transform of the function e^{-as}/s, $a > 0$, $s > 0$. Hence deduce the value of $F_S^{-1}(1/s)$.*

Sol. $$F_S^{-1}\left(\frac{e^{-as}}{s}\right) = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s}\sin sx\, ds = f(x), \text{ say}$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s}\sin sx\, ds \qquad \text{...(1)}$$

Differentiating w.r.t. x, we get

$$\frac{df}{dx} = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s} . s\cos sx\, ds = \frac{2}{\pi}\int_0^\infty e^{-as}\cos sx\, ds$$

$$= \frac{2}{\pi}\cdot\frac{a}{a^2+x^2} \qquad \left(\because \int_0^\infty e^{-ax}\cos bx\, dx = \frac{a}{a^2+b^2}\right)$$

Integrating w.r.t. x, we get

$$f(x) = \frac{2a}{\pi}.\frac{1}{a}\tan^{-1}\frac{x}{a} + c = \frac{2}{\pi}\tan^{-1}\frac{x}{a} + c \qquad \text{...(2)}$$

Putting $x = 0$ in (1) and (2) and equating, we get

$$\frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s}\sin 0\, ds = \frac{2}{\pi}\tan^{-1}\frac{0}{a} + c$$

$$\Rightarrow \quad \frac{2}{\pi} . 0 = 0 + c \quad \Rightarrow \quad c = 0$$

$$\therefore \quad (2) \quad \Rightarrow \quad f(x) = \frac{2}{\pi}\tan^{-1}\frac{x}{a}$$

$$\therefore \quad F_S^{-1}\left(\frac{e^{-as}}{s}\right) = \mathbf{\frac{2}{\pi}\tan^{-1}\frac{x}{a}}.$$

Let $a = 0$.

$$\therefore \quad F_S^{-1}\left(\frac{e^{-0}}{s}\right) = \frac{2}{\pi}\tan^{-1}\frac{x}{0} \quad \text{or} \quad F_S^{-1}\left(\frac{1}{s}\right) = \frac{2}{\pi}\cdot\frac{\pi}{2} = \mathbf{1}.$$

Example 4. *Find the inverse Fourier cosine transform of the function $\frac{1}{1+s^2}$.*

Sol. $$F_C^{-1}\left(\frac{1}{1+s^2}\right) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+s^2}\cos sx\, ds = f(x), \text{ say}$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{\cos sx}{1+s^2}\, ds \qquad \text{...(1)}$$

Differentiating w.r.t. x, we get

$$\frac{df}{dx} = \frac{2}{\pi}\int_0^\infty \frac{-s \sin sx}{1+s^2}\, ds$$

We have $$-\frac{s}{1+s^2} = -\frac{s^2}{s(1+s^2)} = -\frac{(1+s^2)-1}{s(1+s^2)} = -\frac{1}{s} + \frac{1}{s(1+s^2)}$$

$$\therefore \qquad \frac{df}{dx} = \frac{2}{\pi}\int_0^\infty \left(-\frac{1}{s} + \frac{1}{s(1+s^2)}\right) \sin sx\, ds$$

$$= -\frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s}\, ds + \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}\, ds$$

$$= -\frac{2}{\pi}\cdot\frac{\pi}{2} + \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}\, ds \qquad \text{(Assuming } x > 0\text{)}$$

$$\therefore \qquad \frac{df}{dx} = -1 + \frac{2}{\pi}\int_0^\infty \frac{\sin sx}{s(1+s^2)}\, ds \qquad ...(2)$$

Differentiating again w.r.t. x, we get

$$\frac{d^2 f}{dx^2} = 0 + \frac{2}{\pi}\int_0^\infty \frac{s \cos sx}{s(1+s^2)}\, ds = \frac{2}{\pi}\int_0^\infty \frac{\cos sx}{1+s^2}\, ds = f(x) \qquad \text{(By using (1))}$$

$$\Rightarrow \qquad \frac{d^2 f}{dx^2} - f(x) = 0 \quad \Rightarrow \quad (D^2 - 1)f = 0$$

$$\therefore \qquad f(x) = c_1 e^{1.x} + c_2 e^{-1.x} = c_1 e^x + c_2 e^{-x} \qquad ...(3)$$

$$\Rightarrow \qquad \frac{df}{dx} = c_1 e^x - c_2 e^{-x} \qquad ...(4)$$

Putting $x = 0$ in (1) and (3) and equating, we get

$$\frac{2}{\pi}\int_0^\infty \frac{\cos 0}{1+s^2}\, ds = c_1 e^0 + c_2 e^0$$

$$\Rightarrow \qquad \frac{2}{\pi}\tan^{-1} s \Big|_0^\infty = c_1 + c_2 \quad \Rightarrow \quad \frac{2}{\pi}\left(\frac{\pi}{2} - 0\right) = c_1 + c_2$$

$$\Rightarrow \qquad c_1 + c_2 = 1 \qquad ...(5)$$

Putting $x = 0$ in (2) and (4) and equating, we get

$$-1 + \frac{2}{\pi}\int_0^\infty \frac{\sin 0}{s(1+s^2)}\, ds = c_1 e^0 - c_2 e^0$$

$$\Rightarrow \qquad -1 + \frac{2}{\pi}\,.\,0 = c_1 - c_2 \quad \Rightarrow \quad c_1 - c_2 = -1 \qquad ...(6)$$

Solving (5) and (6), we get $c_1 = 0$, $c_2 = 1$

$$\therefore \qquad f(x) = 0.e^{x} + 1 \,.\, e^{-x} = e^{-x}$$

$$\therefore \qquad F_C^{-1}\left(\frac{1}{1+s^2}\right) = \mathbf{e^{-x},\ x > 0.}$$

TEST YOUR KNOWLEDGE

1. Find the inverse Fourier sine and cosine transforms of the function $e^{-\pi s}$.
2. Find the function $F_C^{-1}\left(\frac{\sin 7s}{s}\right)$.
3. Find the function $f(x)$ if $F_C(f(x)) = \begin{cases} \frac{4-s}{4\pi}, & \text{if } 0 < s < 4 \\ 0, & \text{if } \quad s \ge 4. \end{cases}$
4. Find the inverse Fourier sine transform of the function e^{-7s}/s, $s > 0$.
5. Find the inverse Fourier sine transform of the function $\frac{s}{1+s^2}$.

Answers

1. $F_S^{-1}(e^{-\pi s}) = \frac{2x}{\pi(\pi^2 + x^2)}$, $F_C^{-1}(e^{-\pi s}) = \frac{2}{\pi^2 + x^2}$
2. 1 if $0 < x < 7$ and 0 if $x > 7$
3. $\frac{\sin^2 2x}{\pi^2 x^2}$ **4.** $\frac{2}{\pi}\tan^{-1}\frac{x}{7}$ **5.** e^{-x}.

TYPE III. Problems Based on Integrals and Integral Equations

ILLUSTRATIVE EXAMPLES

Example 1. *Show that the* **Laplace integral** $\int_0^\infty \frac{s \sin sx}{a^2 + s^2}\, ds$, *$x > 0$, $a > 0$ is equal to* $\frac{\pi}{2} e^{-ax}$.

Sol. Let $\qquad f(x) = e^{-ax}, \quad x > 0,\ a > 0.$ (Note this step)

$$\therefore \qquad F_S(f(x)) = \int_0^\infty f(x) \sin sx\, dx = \int_0^\infty e^{-ax} \sin sx\, dx$$

$$= \frac{s}{a^2 + s^2} \qquad \left(\text{Using } \int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2 + b^2}\right)$$

$$\therefore \qquad f(x) = F_S^{-1}\left(\frac{s}{a^2 + s^2}\right)$$ (Note that $f(x)$ is defined only for $x > 0$)

$$\Rightarrow \qquad e^{-ax} = \frac{2}{\pi}\int_0^\infty \frac{s}{a^2 + s^2} \sin sx\, ds$$

$$\Rightarrow \qquad \int_0^\infty \frac{s \sin sx}{a^2 + s^2}\, ds = \frac{\pi}{\mathbf{2}}\mathbf{e^{-ax},\ x > 0,\ a > 0.}$$

Example 2. *Show that :*

$$\int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin x\lambda \, d\lambda = \begin{cases} \pi/2\,, & \text{if } 0 < x < \pi \\ 0\,, & \text{if } \quad x > \pi \end{cases}.$$

Sol. Let
$$f(x) = \begin{cases} \pi/2\,, & \text{if } 0 < x < \pi \\ 0\,, & \text{if } \quad x > \pi \end{cases}.$$

$$\therefore \qquad F_S(f(x)) = \int_0^\infty f(x) \sin sx \, dx$$

$$= \int_0^\pi f(x) \sin sx \, dx + \int_\pi^\infty f(x) \sin sx \, dx$$

$$= \int_0^\pi \frac{\pi}{2} \sin sx \, dx + \int_\pi^\infty 0 \,.\, \sin sx \, dx$$

$$= \frac{\pi}{2}\left(-\frac{\cos sx}{s}\right)\Bigg|_0^\pi + 0$$

$$= \frac{\pi}{2s}\,(-\cos \pi s + \cos 0) = \frac{\pi(1-\cos \pi s)}{2s}$$

$$\therefore \qquad f(x) = F_S^{-1}(F_S(f(x)) = F_S^{-1}\left(\frac{\pi(1-\cos \pi s)}{2s}\right) = \frac{2}{\pi}\int_0^\infty \frac{\pi(1-\cos \pi s)}{2s} \sin sx \, ds$$

$$= \int_0^\infty \frac{1-\cos \pi s}{s} \sin sx \, ds = \int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin \lambda x \, d\lambda$$

(Changing s by λ)

$$\therefore \qquad \int_0^\infty \frac{1-\cos \pi\lambda}{\lambda} \sin \lambda x \, d\lambda = f(x) = \begin{cases} \boldsymbol{\pi/2}\,, & \textbf{if} \quad \mathbf{0 < x < \pi} \\ \mathbf{0}\,, & \textbf{if} \quad \mathbf{x > \pi} \end{cases}.$$

Example 3. *Using the Fourier sine transform of the function $e^{-ax} - e^{-bx}$, $a, b > 0, x > 0$, show that :*

$$\int_0^\infty \frac{u \sin ux}{(u^2+a^2)(u^2+b^2)} \, du = \frac{\pi(e^{-ax} - e^{-bx})}{2(b^2-a^2)}, \ x > 0.$$

Sol. Let $f(x) = e^{-ax} - e^{-bx}$, $x > 0$.

$$\therefore \qquad \bar{f}_S(s) = \int_0^\infty f(x) \sin sx \, dx = \int_0^\infty (e^{-ax} - e^{-bx}) \sin sx \, dx$$

$$= \int_0^\infty e^{-ax} \sin sx \, dx - \int_0^\infty e^{-bx} \sin sx \, dx$$

$$= \frac{s}{a^2+s^2} - \frac{s}{b^2+s^2} \qquad \left(\text{Using } \int_0^\infty e^{-ax} \sin bx \, dx = \frac{b}{a^2+b^2}\right)$$

$$= \frac{s(b^2 + s^2 - a^2 - s^2)}{(a^2 + s^2)(b^2 + s^2)} = \frac{s(b^2 - a^2)}{(a^2 + s^2)(b^2 + s^2)}$$

$$\therefore \qquad \bar{f}_S(s) = \frac{s(b^2 - a^2)}{(a^2 + s^2)(b^2 + s^2)}$$

Since the function $f(x)$ is defined only for $(0, \infty)$, we have $F_S^{-1}(\bar{f}_S(s)) = f(x)$.

$$\therefore \qquad \frac{2}{\pi}\int_0^\infty \bar{f}_S(s) \sin sx \, ds = f(x)$$

$$\Rightarrow \qquad \frac{2}{\pi}\int_0^\infty \frac{s(b^2 - a^2)}{(a^2 + s^2)(b^2 + s^2)} \sin sx \, ds = f(x)$$

$$\Rightarrow \qquad \frac{2(b^2 - a^2)}{\pi}\int_0^\infty \frac{s \sin sx}{(a^2 + s^2)(b^2 + s^2)} ds = e^{-ax} - e^{-bx}$$

$$\Rightarrow \qquad \int_0^\infty \frac{s \sin sx}{(a^2 + s^2)(b^2 + s^2)} ds = \frac{\pi(e^{-ax} - e^{-bx})}{2(b^2 - a^2)}$$

$$\therefore \qquad \int_0^\infty \frac{u \sin ux}{(a^2 + u^2)(b^2 + u^2)} du = \boldsymbol{\frac{\pi(e^{-ax} - e^{-bx})}{2(b^2 - a^2)}, \; x > 0.} \qquad \text{(Replacing } s \text{ by } u)$$

Example 4. *Using the Fourier cosine transform of $e^{-x} \cos x$, $x > 0$, show that* $\int_0^\infty \frac{(u^2 + 2)\cos ux}{u^4 + 4} du = \frac{\pi}{2} e^{-x} \cos x, \; x > 0.$

Sol. Let $\qquad f(x) = e^{-x} \cos x, \quad x > 0.$

$$\therefore \qquad \bar{f}_C(s) = \int_0^\infty f(x) \cos sx \, dx = \int_0^\infty e^{-x} \cos x \cos sx \, dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x} (2 \cos sx \cos x) \, dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x} (\cos (s+1)x + \cos (s-1)x) \, dx$$

$$= \frac{1}{2}\int_0^\infty e^{-x} \cos (s+1)x \, dx + \frac{1}{2}\int_0^\infty e^{-x} \cos (s-1)x \, dx$$

$$= \frac{1}{2} \cdot \frac{1}{1 + (s+1)^2} + \frac{1}{2} \cdot \frac{1}{1 + (s-1)^2}$$

$$\left(\text{Using } \int_0^\infty e^{-ax} \cos bx \, dx = \frac{a}{a^2 + b^2} \right)$$

$$= \frac{1}{2}\left[\frac{1}{s^2+2s+2} + \frac{1}{s^2-2s+2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2-2s+2+s^2+2s+2}{(s^2+2)^2-4s^2}\right] = \frac{1}{2}\left[\frac{2s^2+4}{s^4+4}\right] = \frac{s^2+2}{s^4+4}$$

$\therefore$ $$\bar{f}_C(s) = \frac{s^2+2}{s^4+4}$$

Since the function $f(x)$ is defined only for $(0, \infty)$, we have $F_C^{-1}(\bar{f}_C(s)) = f(x)$.

$\therefore$ $$\frac{2}{\pi}\int_0^\infty \bar{f}_C(s) \cos sx \, ds = f(x)$$

$\Rightarrow$ $$\frac{2}{\pi}\int_0^\infty \frac{s^2+2}{s^4+4} \cos sx \, ds = f(x)$$

$\Rightarrow$ $$\int_0^\infty \frac{(s^2+2)\cos sx}{s^4+4} ds = \frac{\pi}{2} \cdot e^{-x} \cos x$$

$\therefore$ $$\int_0^\infty \frac{(u^2+2)\cos ux}{u^4+4} du = \frac{\pi}{2} \, \mathbf{e^{-x} \cos x, x > 0.}$$ (Replacing s by u)

Example 5. *Solve the integral equation :*

$$\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}.$$

Sol. We have $$\int_0^\infty f(x) \cos \lambda x \, dx = e^{-\lambda}.$$

$\Rightarrow$ $$\int_0^\infty f(x) \cos sx \, dx = e^{-s}$$

$\Rightarrow$ $$F_C(f(x)) = e^{-s}$$

$\therefore$ $$f(x) = F_C^{-1}(e^{-s}) = \frac{2}{\pi}\int_0^\pi e^{-s} \cos sx \, ds$$

$$= \frac{2}{\pi}\frac{1}{1+x^2} \qquad \left(\because \int_0^\infty e^{-ax}\cos bx \, dx = \frac{a}{a^2+b^2}\right)$$

$\therefore$ $$f(x) = \frac{\mathbf{2}}{\pi(\mathbf{1}+\mathbf{x^2})}.$$

Example 6. *Solve the integral equation :*

$$\int_0^\infty f(x) \sin sx \, dx = \begin{cases} 1, & \text{if} \quad 0 \le s < 1 \\ 2, & \text{if} \quad 1 \le s < 2 \\ 0, & \text{if} \quad s > 2 \end{cases}.$$

Sol. We have $\displaystyle\int_0^\infty f(x)\sin sx\,dx = \begin{cases} 1, & \text{if } 0 \le s < 1 \\ 2, & \text{if } 1 \le s < 2 \\ 0, & \text{if } s > 2 \end{cases}$.

$$\Rightarrow \qquad F_S(f(x)) = \begin{cases} 1, & \text{if } 0 \le s < 1 \\ 2, & \text{if } 1 \le s < 2 \\ 0, & \text{if } s > 2 \end{cases}$$

$$\therefore \qquad f(x) = F_S^{-1}(F_S(f(x))) = \frac{2}{\pi}\int_0^\infty F_S(f(x))\sin sx\,ds$$

$$= \frac{2}{\pi}\left[\int_0^1 1.\sin sx\,ds + \int_1^2 2.\sin sx\,ds + \int_2^\infty 0.\sin sx\,ds\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos sx}{x}\Big|_0^1 + \frac{2(-\cos sx)}{x}\Big|_1^2 + 0\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos x}{x} + \frac{1}{x} - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{x} + \frac{\cos x}{x} - \frac{2\cos 2x}{x}\right] = \boldsymbol{\frac{2}{\pi x}(1 + \cos x - 2\cos 2x).}$$

TEST YOUR KNOWLEDGE

1. Show that the *Laplace integral* $\displaystyle\int_0^\infty \frac{\cos sx}{a^2+s^2}\,ds$, $x > 0$, $a > 0$ is equal to $\dfrac{\pi}{2a}\,e^{-ax}$.

2. Show that $\displaystyle\int_0^\infty \frac{\cos\lambda x}{\lambda^2+1}\,d\lambda = \frac{\pi}{2}\,e^{-x}, \quad x \ge 0.$

3. Find the Fourier sine transform of the function $e^{-|x|}$, $x > 0$. Hence show that :

$$\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m},\ m > 0.$$

4. Using the Fourier cosine transform of the function $f(x) = \begin{cases} 1, & \text{if } 0 < x < 1 \\ 0, & \text{if } x > 1 \end{cases}$, evaluate the integral

$$\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\,d\lambda.$$

5. Solve the integral equation : $\displaystyle\int_0^\infty f(x)\sin\alpha x\,dx = \begin{cases} 1-\alpha, & \text{if } 0 \le \alpha \le 1 \\ 0, & \text{if } \alpha > 1 \end{cases}$.

6. Solve the integral equation : $\displaystyle\int_0^\infty f(x)\cos\lambda x\,dx = \begin{cases} 1-\lambda, & \text{if } 0 \le \lambda \le 1 \\ 0, & \text{if } \lambda > 1 \end{cases}$.

Hence deduce that $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$.