

Triple Integrals in Cylindrical Coordinates

التكاملات الثلاثية بالإحداثيات الاسطوانية

Example 2: By using the **cylindrical coordinate**, find the volume of the spherical (الكرة) with radius 2, i.e., $x^2 + y^2 + z^2 = 4$.

Solution:

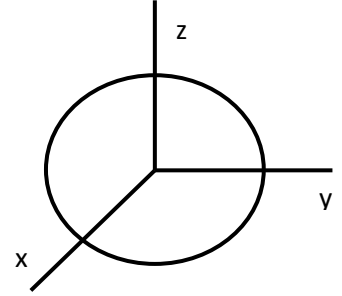
$$x^2 + y^2 + z^2 = 4 \Rightarrow z^2 = 4 - x^2 - y^2 \Rightarrow z^2 = 4 - r^2 \Rightarrow z = \pm\sqrt{4 - r^2}$$

$$\int_0^{2\pi} \int_0^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 z \Big|_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r \, dr \, d\theta =$$

$$= \int_0^{2\pi} \int_0^2 \left[\sqrt{4-r^2} - \left[-\sqrt{4-r^2} \right] \right] r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 2\sqrt{4-r^2} \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^2 -2r\sqrt{4-r^2} \, dr \, d\theta = - \int_0^{2\pi} \left[\frac{(4-r^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^2 d\theta = - \frac{2}{3} \int_0^{2\pi} -[4]^{\frac{3}{2}} d\theta = \frac{16}{3} \int_0^{2\pi} d\theta$$

$$= \frac{16}{3} \theta \Big|_0^{2\pi} = \frac{32\pi}{3} \text{ unit}^3$$



Second method

$$8 \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{\sqrt{4-r^2}} dz \, r \, dr \, d\theta = 8 \int_0^{\frac{\pi}{2}} \int_0^2 z \Big|_0^{\sqrt{4-r^2}} r \, dr \, d\theta = 8 \int_0^{\frac{\pi}{2}} \int_0^2 \sqrt{4-r^2} \, r \, dr \, d\theta =$$

$$= 8 \int_0^{\frac{\pi}{2}} \left[\frac{(4-r^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^2 d\theta = - \frac{8}{3} \int_0^{\frac{\pi}{2}} \left[(0)^{\frac{3}{2}} - 4^{\frac{3}{2}} \right] d\theta = - \frac{8}{3} \int_0^{\frac{\pi}{2}} -8 \, d\theta = \frac{64}{3} \theta \Big|_0^{\frac{\pi}{2}} = \frac{32}{3} \pi$$

Example 3: By using the **cylindrical coordinate**, find the volume bounded by paraboloid $z = x^2 + y^2$ and the plane $z = 9$.

Solution:

$$z = x^2 + y^2 \Rightarrow z = r^2$$

$$\begin{aligned} 4 \int_0^{\frac{\pi}{2}} \int_0^3 \int_{r^2}^9 dz \, r \, dr \, d\theta &= 4 \int_0^{\frac{\pi}{2}} \int_0^3 z \Big|_{r^2}^9 r \, dr \, d\theta = 4 \int_0^{\frac{\pi}{2}} \int_0^3 (9 - r^2) r \, dr \, d\theta \\ &= 4 \int_0^{\frac{\pi}{2}} \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_0^3 d\theta = 4 \int_0^{\frac{\pi}{2}} \left(\frac{81}{2} - \frac{81}{4} \right) d\theta \\ &= 4 \int_0^{\frac{\pi}{2}} \frac{81}{4} d\theta = 81 \int_0^{\frac{\pi}{2}} d\theta = 81 \left[\theta \right]_0^{\frac{\pi}{2}} = \frac{81}{2} \pi \end{aligned}$$

Second method

$$\begin{aligned} \int_0^{2\pi} \int_0^3 \int_{r^2}^9 dz \, r \, dr \, d\theta &= \int_0^{2\pi} \int_0^3 z \Big|_{r^2}^9 r \, dr \, d\theta = \int_0^{2\pi} \int_0^3 (9 - r^2) r \, dr \, d\theta \\ &= \int_0^{2\pi} \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_0^3 d\theta = \int_0^{2\pi} \left(\frac{81}{2} - \frac{81}{4} \right) d\theta \\ &= \int_0^{2\pi} \frac{81}{4} d\theta = \frac{81}{4} \int_0^{2\pi} d\theta = \frac{81}{4} \left[\theta \right]_0^{2\pi} = \frac{81}{2} \pi \end{aligned}$$

Example 4: By using the **cylindrical coordinate**, find the volume of the solid region cut from the spherical (الكرة) $x^2 + y^2 + z^2 = 16$ by cylindrical $x^2 + y^2 = 4$.

Solution:

$$x^2 + y^2 + z^2 = 16 \Rightarrow z^2 = 16 - x^2 - y^2 \Rightarrow z = \pm \sqrt{16 - r^2}$$

$$\begin{aligned} 8 \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^{\sqrt{16-r^2}} dz \, r \, dr \, d\theta &= 8 \int_0^{\frac{\pi}{2}} \int_0^2 z \Big|_0^{\sqrt{16-r^2}} r \, dr \, d\theta = 8 \int_0^{\frac{\pi}{2}} \int_0^2 \sqrt{16-r^2} r \, dr \, d\theta = \\ &= 8 \int_0^{\frac{\pi}{2}} \left[-\frac{(16-r^2)^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^2 d\theta = -\frac{16}{3} \int_0^{\frac{\pi}{2}} \left[(12)^{\frac{3}{2}} - (16)^{\frac{3}{2}} \right] d\theta = \\ &= -\frac{16}{3} \left[(12)^{\frac{3}{2}} \theta - (16)^{\frac{3}{2}} \theta \right]_0^{\frac{\pi}{2}} = -\frac{16}{3} \left[(12)^{\frac{3}{2}} - (16)^{\frac{3}{2}} \right] \pi \end{aligned}$$

Example 5: Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder (اسطوانة) $x^2 + y^2 = 4$, bounded above by the paraboloid (الجسم) $z = x^2 - y^2$, and bounded below by the xy -plane

Solution:

$$\begin{aligned} M &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} \delta \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 Z|_0^{r^2} r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 r^3 \, dr \, d\theta = \int_0^{2\pi} \left[\frac{r^4}{4} \right]_0^2 d\theta = \int_0^{2\pi} 4 \, d\theta = 4\theta|_0^{2\pi} = 8\pi \text{ mass} \end{aligned}$$

$$\begin{aligned} M_{yz} &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} x \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (r \cos \theta) Z|_0^{r^2} r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 (r \cos \theta) r^3 \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \cos \theta \, r^4 \, dr \, d\theta \\ &= \int_0^{2\pi} \cos \theta \left[\frac{r^5}{5} \right]_0^2 d\theta = \frac{32}{5} \int_0^{2\pi} \cos \theta \, d\theta = \frac{32}{5} \sin \theta|_0^{2\pi} = 0, \quad \therefore M_{yz} = 0 \end{aligned}$$

$$\begin{aligned} M_{xz} &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} y \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 (r \sin \theta) Z|_0^{r^2} r \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^2 (r \sin \theta) r^3 \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \sin \theta \, r^4 \, dr \, d\theta \\ &= \int_0^{2\pi} \sin \theta \left[\frac{r^5}{5} \right]_0^2 d\theta = \frac{32}{5} \int_0^{2\pi} \sin \theta \, d\theta = -\frac{32}{5} \cos \theta|_0^{2\pi} = 0, \quad \therefore M_{xz} = 0 \end{aligned}$$

$$\begin{aligned} M_{xy} &= \int_0^{2\pi} \int_0^2 \int_0^{r^2} z \, dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 \left[\frac{z^2}{2} \right]_0^{r^2} r \, dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \int_0^2 r^5 \, dr \, d\theta \\ &= \frac{1}{2} \int_0^{2\pi} \left[\frac{r^6}{6} \right]_0^2 d\theta = \frac{1}{12} \int_0^{2\pi} 64 \, d\theta = \frac{64}{12} \int_0^{2\pi} d\theta = \frac{32\pi}{3}, \quad \therefore M_{xy} = \frac{32\pi}{3} \end{aligned}$$

Center of mass is

$$\bar{X} = \frac{M_{yz}}{M} = \frac{0}{8\pi} = 0, \quad \bar{Y} = \frac{M_{xz}}{M} = \frac{0}{8\pi} = 0, \quad \bar{Z} = \frac{M_{xy}}{M} = \frac{\frac{32\pi}{3}}{8\pi} = \frac{4}{3}$$

Q.1: Find the mas, first moments, and center of mass of triangular with vertices $(0,1), (1,1), (1,2)$, and the density at (x, y) is $\delta(x, y) = 1$.

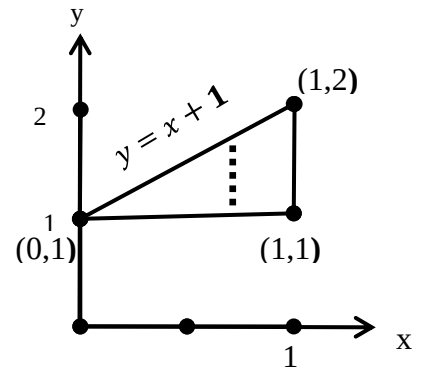
Solution:

$$\frac{x-x_0}{x_1-x_0} = \frac{y-y_0}{y_1-y_0} \Rightarrow \frac{x-0}{1-0} = \frac{y-1}{2-1} \Rightarrow y = x + 1$$

Mass

$$M = \iint_R \delta(x, y) dA$$

$$M = \int_0^1 \int_1^{x+1} dy dx = \int_0^1 y \Big|_1^{x+1} dx = \int_0^1 (x+1-1) dx = \int_0^1 x dx = \frac{x^2}{2} \Big|_0^1 = \frac{1}{2}$$



First moments

$$M_x = \int_0^1 \int_1^{x+1} y dy dx = \int_0^1 \frac{y^2}{2} \Big|_1^{x+1} dx = \int_0^1 \left(\frac{(x+1)^2}{2} - \frac{1}{2} \right) dx = \frac{(x+1)^3}{6} - \frac{x}{2} \Big|_0^1 =$$

$$\frac{8}{6} - \frac{1}{2} - \left[\frac{1}{6} \right] = \frac{7}{6} - \frac{1}{2} = \frac{7-3}{6} = \frac{4}{6} = \frac{2}{3}$$

$$M_y = \int_0^1 \int_1^{x+1} x dy dx = \int_0^1 x y \Big|_1^{x+1} dx = \int_0^1 x (x+1-1) dx = \int_0^1 x^2 dx = \frac{x^3}{3} \Big|_0^1 =$$

$$= \frac{1}{3}$$

Center of mass

$$\bar{X} = \frac{M_y}{M} = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}, \quad \bar{Y} = \frac{M_x}{M} = \frac{\frac{2}{3}}{\frac{1}{2}} = \frac{4}{3}$$

Q.2: By using the **cylindrical coordinate**, find the volume of the solid bounded by the cylinder $x^2 + y^2 = 4$ and the two planes $z = -2$ and $z = 2$.

Solution:

$$V = \int_0^{2\pi} \int_0^2 \int_{-2}^2 dz \, r \, dr \, d\theta = \int_0^{2\pi} \int_0^2 z]_{-2}^2 r \, dr \, d\theta$$

$$\int_0^{2\pi} \int_0^2 4 r \, dr \, d\theta = \int_0^{2\pi} 2r^2]_0^2 d\theta = \int_0^{2\pi} 8 d\theta = 16\pi \text{ unit}^3$$

Second method

$$V = 2 \int_0^{2\pi} \int_0^2 \int_0^2 dz \, r \, dr \, d\theta = 2 \int_0^{2\pi} \int_0^2 z]_0^2 r \, dr \, d\theta =$$

$$= 2 \int_0^{2\pi} \int_0^2 2r \, dr \, d\theta = 2 \int_0^{2\pi} r^2]_0^2 d\theta =$$

$$= 2 \int_0^{2\pi} 4 d\theta = (2) 4\theta]_0^{2\pi} = 16\pi \text{ unit}^3$$

Third method

$$V = 2(4) \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^2 dz \, r \, dr \, d\theta = 8 \int_0^{\frac{\pi}{2}} \int_0^2 z]_0^2 r \, dr \, d\theta =$$

$$= 8 \int_0^{\frac{\pi}{2}} \int_0^2 2r \, dr \, d\theta = 8 \int_0^{\frac{\pi}{2}} r^2]_0^2 d\theta =$$

$$8 \int_0^{\frac{\pi}{2}} 4 d\theta = (8) 4\theta]_0^{\frac{\pi}{2}} = 16\pi \text{ unit}^3$$

