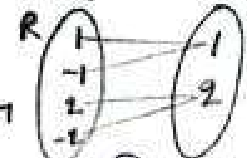
Example:- To prove the converse is not true.

البرهنة × f, g functions
$g \circ f$ bijective
$\Rightarrow f$ is 1-1
g is onto
⇑ العكس

Sol.:- No, since



Let $f: \mathbb{R} \to \mathbb{R}$ s.t. $f(x) = x$

$g: \mathbb{R} \to \mathbb{R}^+$ s.t. $g(x) = x^2$

f is 1-1 and g is onto

نلاحظ ان

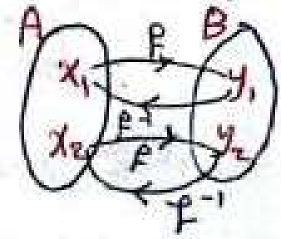
$\to \forall y \in \mathbb{R}^+, \exists x \in \mathbb{R}$ s.t. $(x, y) \in g$ or $y = g(x)$

$(g \circ f)(x) = g(f(x)) = g(x) = x^2$

$-1 \neq 1$
$\Rightarrow g(-1) = g(1)$

$\therefore g \circ f$ is not bijective (not one to one, onto)

Def.:- A function $f: A \to B$ is said to be invertible (معكوس) if $f^{-1}: B \to A$ is a function.



Ex.(1):- let $f(x) = x + 3$, $f: \mathbb{R} \to \mathbb{R}$

$y = x + 3 \Rightarrow x = y - 3$

$f^{-1}(y) = y - 3$ is a function.

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المنطلق له
صورة

$\therefore f$ is invertible.

$y = f(x) \Longleftrightarrow x = f^{-1}(y)$

$f^{-1}(y) = y - 3$
اي قيمة ناخذها لـ y
الناتج ينتمي لـ R

Ex.(2):- $g: \mathbb{R} \to \mathbb{R}$ s.t. $g(x) = x^2$

$y = x^2 \Rightarrow x = \mp\sqrt{y}$

$\therefore g(x)$ is not invertible

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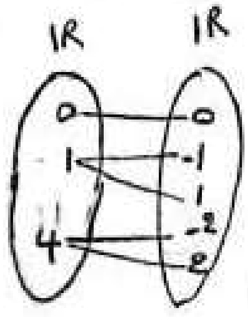


$g = \{(0,0), (1,1), (-1,1), (2,4), (-2,4), \ldots\}$

$g^{-1} = \{(0,0), (1,1), (1,-1), (4,2), (4,-2), \ldots\}$

$g^{-1}(y) = \mp\sqrt{y}$ is not function

Ex. (3):- Let $f: A \to B$, $A = \{1,2,3\}$, $B = \{x,y,z\}$

$$f = \{(1,x), (2,y), (3,z)\}$$

$$f^{-1} = \{(x,1), (y,2), (z,3)\}$$

f is invertible.

Theorem:- Let $f: A \to B$ be an invertible function.

Then ① $f^{-1} \circ f = I_A$ ② $f \circ f^{-1} = I_B$

$f^{-1}(y = f(x))$
$f^{-1}(y) = f^{-1} f(x)$
$f^{-1}(y) = x$

Proof:- ① Let $x \in A$ and $y = f(x)$ then $x = f^{-1}(y)$ من تعريف f^{-1}

thus $[f^{-1} \circ f](x) = f^{-1}(f(x)) = f^{-1}(y) = x = I_A$.

② Let $x \in A$ and $y = f(x)$, then $x = f^{-1}(y)$

thus $[f \circ f^{-1}](y) = f(f^{-1}(y)) = f(x) = y = I_B$.

Theorem:- If $f: A \to B$ is bijective function, then f^{-1} is a bijective function.

Proof:- let $y_1, y_2 \in B$ s.t. $f^{-1}(y_1) = f^{-1}(y_2)$

$$\Rightarrow f(f^{-1}(y_1)) = f(f^{-1}(y_2))$$

$$\Rightarrow (f \circ f^{-1})(y_1) = (f \circ f^{-1})(y_2)$$

$$\Rightarrow I_B(y_1) = I_B(y_2)$$

$$\Rightarrow y_1 = y_2 \quad \therefore f^{-1} \text{ is one to one.}$$

let $y \in B$ $\because$ f is bijective $\Rightarrow$ f is onto.

$\Rightarrow \exists x \in A$ s.t. $y = f(x)$ or $(x,y) \in f$

$\Rightarrow \forall x \in A, \exists y \in B$ s.t. $f^{-1}(y) = x$ or $(y,x) \in f^{-1}$

$\therefore f^{-1}$ is onto $\Rightarrow f^{-1}$ is bijective.

Theorem:- Let $f: A \to B$ and $g: B \to A$ be functions. If $g \circ f = I_A$ and $f \circ g = I_B$ then $f: A \to B$ is bijective and $g = f^{-1} =$ invertible(f).

Proof:- let $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$

$$\Rightarrow g(f(x_1)) = g(f(x_2))$$

$$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$$

$$\Rightarrow I_A x_1 = I_A x_2$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$ is one to one.

if $y \in B$ then $\underline{y} = I_B(y) = (f \circ g)(y) = f(\underline{g(y)})$

if y is any element of B, then $\underline{y} = f(\underline{x})$.

where $\boxed{x = g(y)} \xrightarrow[x = f^{-1}(y)]{\text{لازم نثبت}}$

$\therefore f$ is onto.

$y = f(\underline{g(y)})$

$y = f(\underline{x})$

$\therefore x = g(y)$

we prove $g = f^{-1} \to \begin{matrix} x = g(y) \\ x = f^{-1}(y) \end{matrix}$ and $\begin{matrix} x = f^{-1}(y) \\ x = g(y) \end{matrix}$

$\Rightarrow f(x) = f(g(y)) = (f \circ g)(y) = I_B(y) = y$

$\Rightarrow \boxed{x = f^{-1}(y)}$

$\Leftarrow x = f^{-1}(y) \Rightarrow y = f(x)$

$$\Rightarrow g(y) = g(f(x)) = (g \circ f)(x) = I_A(x) = \underline{x}.$$

thus $\forall y \in B$, $x = f^{-1}(y)$ iff $x = g(y)$ that is $f^{-1}(y) = g(y)$.

by Th. [$f: A \to B$, $g: A \to B$ then $f = g$ iff $f(x) = g(x)$, $\forall x \in A$].

$\therefore f^{-1} = g$.

Theorem:- If $f: A \to B$, $g: B \to C$ and $g \circ f: A \to C$ are function then:-

① if f and g are injective, then $g \circ f$ is injective.

② if f and g are onto, then $g \circ f$ is onto.

③ if f and g are bijective, then $g \circ f$ is bijective.

Proof(1):- $\forall x_1, x_2 \in A$, $(g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow f(x_1) = f(x_2)$ (since g is 1-1)

$\Rightarrow x_1 = x_2$ (since f is 1-1)

$\therefore g \circ f$ is injective.

② $\forall z \in C$, then $\exists y \in B$ s.t. $z = g(y)$ or $(y, z) \in g$ (since g is onto)

since $y \in B$ $\exists x \in A$ s.t. $y = f(x)$ or $(x, y) \in f$ (since f is onto).

that is $\forall z \in C$, $\exists x \in A$ s.t $z = g(f(x)) = (g \circ f)(x) \Rightarrow$

$g \circ f$ is onto.

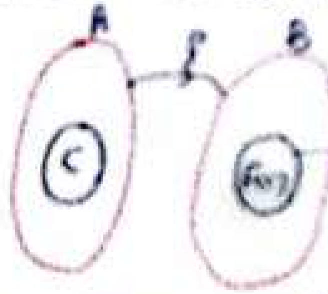
③ From ① and ② we get $g \circ f$ is bijective.

$(x, y) \in f$
$(y, z) \in g$
$\Rightarrow (x, z) \in g \circ f$

Direct Images and Inverse Images under Function

الصورة المباشرة و معكوس الصورة المباشرة تحت تأثير الدالة

Def.: Let $f: A \to B$ be a function, if C is any subset of A, the direct image of C under f, which we write $F(C)$ is the following subset of B.



→ the direct image

الصورة المباشرة

$$F(C) = \{y \in B : \underline{\exists x \in C} \text{ s.t } y = f(x)\}$$

that is, $F(C)$ is the set of all the image of elements in C. هي المجموعة كل صور العناصر الموجودة في المجموعة C

Ex.: $f: Z \longrightarrow R$, $f(x) = x^2 + 2$

$C = \{-3, -1, 0, 1, 3\} \longrightarrow$ مجموعة جزئية من Z

$F(C) = \{2, 3, 11\} = \{f(-3), f(-1), f(0), f(1), f(3)\}$ صور هذه العناصر

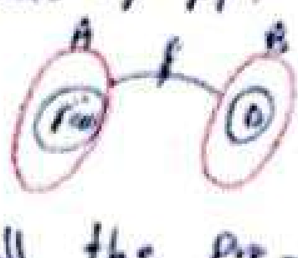
مجموعة جزئية من R

Def.: The inverse image

Let $f: A \to B$ be a function, if D is any subsets of B, the inverse image of D under f, which we write $F^{-1}(D)$, is the following subsets of A.

صورها في D

$$F^{-1}(D) = \{x \in A : f(x) \in D\}$$



That is, $F^{-1}(D)$ is the sets of all the pre-image of element in D.

مجموعة كل معكوس الصور الموجودة في D

يعني D صور العناصر من أصل الانفرس

علشان بيطلع معكوس الصور