Chapter Five
الاسـتـمـراريـة فـي الـفـضـاءات الـتـبـولـوجـيـة
The Continuity on Topological Spaces

Definition (5.1): We say that the function $f: (X, \tau) \to (X^*, \tau^*)$ is continuous at a point $a \in X$ iff for every open set $G^* \ni f(a), \exists^{open} G \ni a, f(G) \subset G^*$.

That is: f continuous at $a \in X$ iff $\forall^{open} G^* \ni f(a), \exists^{open} G \ni a, f(G) \subset G^*$

Example (5.1): Let $X = \{a, b, c, d, e\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, and

$X^* = \{x, y, z, u\}$, $\tau^* = \{\emptyset, \{x\}, \{y\}, \{x, y\}, \{x, y, z\}, X\}$.

Let $f: X \to X^*$ be defined as follows:

$f(a) = x, f(b) = f(c) = y, f(d) = z, f(e) = z$

Determine whether f is continuous at the point a, b, c, d, e

Solution: $a \in X, f(a) = x$

The open sets G^* containing $f(a) = x$ are $\{x\}, \{x, y\}, \{x, y, z\}, X^*$

The open sets G containing a are $\{a\}, \{a, b\}, X$

We have $f(\{a\}) = \{f(a)\} = \{x\} \subset G^*, \forall G^* \ni f(a)$

$\Rightarrow \exists^{open} G = \{a\} \ni a, \forall^{open} G \ni f(a), f(G) \subset G^*, \forall G^* \ni f(a)$

$\Rightarrow f$ is continuous at $a \in X$

$b \in X, f(b) = y$

The open sets G^* containing $f(b) = y$ are $\{y\}, \{x, y\}, \{x, y, z\}, X^*$

The open sets G containing b are $\{b\}, \{a, b\}, X$

We have $f(\{b\}) = \{f(b)\} = \{y\} \subset G^*, \forall G^* \ni f(b)$

$\Rightarrow \exists^{open} G = \{b\} \ni b, \forall^{open} G \ni f(b), f(G) \subset G^*, \forall G^* \ni f(b)$

$\Rightarrow f$ is continuous at $b \in X$

$c \in X, f(c) = y$

The open sets G^* containing $f(c) = y$ are $\{y\}, \{x, y\}, \{x, y, z\}, X^*$

The open set G containing c is X

We have $f(\{c\}) = \{f(c)\} = \{y\} \subset G^*, \forall\, G^* \ni f(c)$

$\Rightarrow \exists^{open}\, G = \{c\} \ni c, f(G) \not\subset G^*$

$f(x) = \{x, y, z\} \not\subset G^*$

$[\forall\, G^* \ni y, \exists\, X \ni c, f(x) \not\subset G^*]$

$d \in X, f(d) = z$

The open sets G^* containing $f(d) = z$ are $\{x, y, z\}, X^*$

The open sets G containing d is X

We have $f(\{d\}) = \{f(d)\} = \{z\} \subset G^*, \forall\, G^* \ni f(d)$

$\Rightarrow \exists^{open}\, G = \{d\} \ni d, \forall^{open}\, G \ni f(d), f(G) \subset G^*, \forall\, G^* \ni f(d)$

$\Rightarrow f$ is continuous at $d \in X$

$e \in X, f(e) = z$

The open sets G^* containing $f(e) = z$ are $\{x, y, z\}, X^*$

The open set G containing e is X

We have $f(\{e\}) = \{f(e)\} = \{z\} \subset G^*, \forall\, G^* \ni f(e)$

$\Rightarrow \exists^{open}\, G = \{e\} \ni e, \forall^{open}\, G \ni f(e), f(G) \subset G^*, \forall\, G^* \ni f(e)$

$\Rightarrow f$ is continuous at $e \in X$

Remark (5.1): If f is continuous at each point $x \in X$. Then we say that f is continuous on X.

Theorem (5.1): (The Fundamental Theorem of Continuity)

If $f: (X, \tau) \to (X^*, \tau^*)$. Then the continuity of f on X is equivalent to each of the following conditions:

(1) The inerse image of each open set in X^* is open in X.

(2) The inverse image of each closed set in X^* is closed in X.

(3) $f(\bar{A}) \subset \overline{f(A)}, \forall\, A \subset X$.

Proof: Continuity of $f \leftrightarrow f^{-1}(G^*)$ is open in $X, \forall^{open}\, G^* \subset X^*$

Continuity of $f \leftrightarrow f^{-1}(G^*)$ is closed in $X, \forall^{closed}\, G^* \subset X^*$

Suppose that f is continuous on X

We need to show that $f^{-1}(G^*)$ is open in X, $\forall^{open} G^* \subset X^*$

Let $G^* \subset X^*$ be any open we have to prove $f^{-1}(G^*)$ is open

Let $x \in f^{-1}(G^*)$ be any point

$\Rightarrow f(x) \in G^*$, but $G^* \subset X^*$ is given open and $f(x) \in G^*$

Since f is continuous

$\Rightarrow \forall^{open} G^* \ni f(x), \exists^{open} G \ni x, f(G) \subset G^*$

$\Rightarrow f(x) \in f(G) \subset G^* \Rightarrow x \in G \subset f^{-1}(G^*)$

$\Rightarrow \forall\ x \in f^{-1}(G^*), \exists^{open} G \ni x, G \subset f^{-1}(G^*)$

$\Rightarrow f^{-1}(G^*)$ is open in X

Conversly: Suppose that $f^{-1}(G^*)$ is open in X, $\forall^{open} G^* \subset X^*$

We need to show that f is continuous on X

Let $G^* \subset X^*$ be any open set $\Rightarrow f^{-1}(G^*) \subset X$ is open

But by using the definition of open set we set

$\forall\ x \in f^{-1}(G^*), \exists^{open} G \ni x, x \in G \subset f^{-1}(G^*)$

$\Rightarrow \forall\ x \in f^{-1}(G^*), \exists^{open} G \ni x, f(x) \in f(G) \subset G^* \Rightarrow f(x) \in f(G) \subset G^*$

$\Rightarrow \forall^{open} G^* \subset X^*, \exists^{open} G, f(G) \subset G^* \Rightarrow f$ is continuous on X

Now (1) $\leftrightarrow$ (2)

Suppose the inverse image of each open set in X^* is open in X

We need to show that the inverse image of each closed set in X^* is closed in X

Let $F^* \subset X^*$ be any closed set

$\Rightarrow F^{*^c} = X^* - F^*$ is open set

$\Rightarrow f^{-1}(F^{*^c})$ is open $\Rightarrow \left(f^{-1}(F^*)\right)^c \Rightarrow f^{-1}(F^*)$ is closed in X

Conversely: Suppose that the inverse of image of each closed set in X^* is closed in X.

We need to show that the inverse image of each open set in X^* is open in X

$\Rightarrow f^{-1}(G^{*^c})$ is closed in $X \Rightarrow \left(f^{-1}(G^*)\right)^c$ is closed in X

$\Rightarrow f^{-1}(G^*)$ is open in X

Now we prove $(2) \leftrightarrow (3)$

Suppose that the inverse image of each closed set in X^* is closed in X

We need to show that $f(\bar{A}) \subset \overline{f(A)}, \forall A \subset X$

Since $f(A) = f(A), \forall A \subset X$ and $f(A) \subset \overline{f(A)}$ closed

$\Rightarrow A \subset f^{-1}(\overline{f(A)})$ closed

$\Rightarrow \bar{A} \subset \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)})$

$\Rightarrow \bar{A} \subset f^{-1}(\overline{f(A)}) \Rightarrow f(\bar{A}) \subset \overline{f(A)}$

Conversely: suppose that $f(\bar{A}) \subset \overline{f(A)}, \forall A \subset X$

We need to show that $f^{-1}(F^*)$ is closed in $X, \forall^{closed} F \subset X^*$

Let $F^* \subset X^*$ be any closed set

We need to prove that $f^{-1}(F^*)$ is closed in X

Let $E = f^{-1}(F^*)$

We are given $f(\bar{E}) \subset \overline{f(E)} = \overline{f(f^{-1}(F^*))} = \overline{F^*} = F^* = f(E)$

$\Rightarrow f(\bar{E}) \subset f(E)$ but $E \subset \bar{E} \Rightarrow f(E) \subset f(\bar{E})$

$\Rightarrow f(\bar{E}) = f(E)$

$\Rightarrow f^{-1}(f(\bar{E})) = f^{-1}(f(E))$

$\Rightarrow \bar{E} = E = f^{-1}(F^*) \Rightarrow f^{-1}(F^*)$ is closed in X.