Example (6.6): Let $< f_n >$ be a sequence of function on R defined by

$f_n(x) = \frac{n}{x+n}$, $\forall\, x \in [0, \infty)$, $\forall\, n \in N$

Solution:

Since $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{n}{x+n} = 1$, $\forall\, x \in [0, \infty)$

Thus $f_n(x)$ converges pointwise to $f(x) = 1$.

Definition (6.3): (Uniform Convergence) التقارب المنتظم

We say that the sequence $< f_n >$ **convergent uniformly** to f_0 if

$\forall\, \varepsilon > 0, \exists\, k \in N, k = k(\varepsilon)$ such that $|f_n(x) - f(x)| < \varepsilon$, $\forall\, n > k, \forall\, x \in S$

We write $f_n(x) \overset{U}{\rightarrow} f(x)$.

Example (6.7): Let $< f_n >$ be a sequence of function on R defined by

$f_n(x) = \frac{x}{n}$, $\forall\, x \in R$, $\forall\, n \in N$

Solution:

Let $\varepsilon > 0, \exists\, k \in N$, such that

$$|f_n(x) - f(x)| = \left|\frac{x}{n} - 0\right|$$

$$= \left|\frac{x}{n}\right| < \varepsilon \quad , \quad \forall\, n > k, \quad \forall\, x \in R$$

i.e. $< f_n(x) >=< \frac{x}{n} > \rightarrow \; f(x) = 0$ (zero function)

$\Rightarrow f_n(x) \overset{U}{\rightarrow} f(x)$, where $f(x) = 0$, $\forall\, x \in R$

Or $f_n(x)$ is convergent uniformly to $f(x) = 0$.

Example (6.8): Let $< f_n >$ be a sequence of function on R defined by

$f_n(x) = \frac{nx}{1+nx}$, $\forall\, x \in [0,1]$, $\forall\, n \in N$

Solution: Let $\varepsilon > 0, \exists\, k \in N$, such that

$$|f_n(x) - f(x)| = \left|\frac{nx}{1+nx} - 1\right|$$

$$= \left|\frac{-1}{1+nx}\right| < \varepsilon \quad , \quad \forall\, n > k, \quad \forall\, x \in [0,1]$$

i.e. $< f_n(x) >=< \frac{nx}{1+nx} > \ \rightarrow \ f(x) = 1$ (constant function)

$\Rightarrow \ f_n(x) \overset{U}{\rightarrow} f(x)$, where $f(x) = 1$, $\forall\, x \in R$

Or $f_n(x)$ is convergent uniformly to $f(x) = 1$.

Since $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} \frac{nx}{1+nx} = 1$, $\forall\, x \in [0,1]$

Thus $f_n(x)$ converges pointwise to $f(x) = 1$.

Example (6.9): Let $f_n: [0,1] \rightarrow R$, defined by $f_n(x) = x^n$, $\forall\, x \in [0,1]$, $\forall\, n \in N$, show that the sequence of function $< f_n >$ is not uniformly convergent on $[0,1]$.

Solution:

Since $< f_n(x) >=< x^n > \ \rightarrow \begin{cases} 0 & if \ \ x \in [0,1) \\ 1 & if \qquad x = 1 \end{cases}$

$\Rightarrow \ f_n(x) \overset{U}{\nrightarrow} f(x)$,

Or $f_n(x)$ is not convergent uniformly.

Example (6.10): Let $f_n: [0,1] \rightarrow R$, defined by $f_n(x) = e^{\frac{x}{n}}$, $\forall\, x \in [0,1]$, $\forall\, n \in N$, show that the sequence of function $< f_n >$ is convergent uniformly.

Solution:

Let $\varepsilon > 0$, $\exists\, k \in N$, such that

$$|f_n(x) - f(x)| = \left|e^{\frac{x}{n}} - 1\right| < \varepsilon \quad , \quad \forall\, n > k, \quad \forall\, x \in [0,1]$$

i.e. $< f_n(x) >=< e^{\frac{x}{n}} > \ \rightarrow \ f(x) = 1$ (constant function)

$\Rightarrow \ f_n(x) \overset{U}{\rightarrow} f(x)$, where $f(x) = 1$, $\forall\, x \in [0,1]$

Or $f_n(x)$ is convergent uniformly to $f(x) = 1$.

Example (6.11): Let $< f_n >$ be a sequence of function on R defined by

$f_n(x) = \frac{n}{x+n}$, $\forall\, x \in [0, \infty)$, $\forall\, n \in N$

Solution:

Let $\varepsilon > 0$, $\exists\, k \in N$, such that

$$|f_n(x) - f(x)| = \left|\frac{n}{x+n} - 1\right|$$

$$= \left|\frac{-x}{x+n}\right| < \varepsilon \quad , \quad \forall\, n > k, \quad \forall\, x \in [0, \infty)$$

i.e. $< f_n(x) > = < \frac{n}{x+n} > \;\to\; f(x) = 1$ (constant function)

$\Rightarrow\; f_n(x) \overset{U}{\to} f(x)$, where $f(x) = 1$, $\forall\, x \in R$

Or $f_n(x)$ is convergent uniformly to $f(x) = 1$.

Theorem (6.1):

Let $< f_n >$ be a sequence of functions convergent uniformly to f, then

(1) If f_n is bounded on S , $\forall\, n$, then f is bounded.

(2) If f_n is continuous on S , $\forall\, n$, then f is continuous.

Definition (6.4): A **series** of functions $\sum_{n=1}^{\infty} f_n = f_1 + f_2 + f_3 + \cdots$ where f_n is a real function

$\forall\; x \in X$, $S_1(x) = f_1(x)$

$$S_2(x) = f_1(x) + f_2(x)$$

$$\vdots$$

$$S_n(x) = f_1(x) + f_2(x) + \cdots + f_n(x)$$

$$\vdots$$

We say $\sum f_n$ convergent at $x = x_0$ if $< S_n(x) >$ is convergent. If $S_n \to S$ then We write $\sum_{n=1}^{\infty} f_n(x) = S$

Definition (6.5): (Power Series) **متسلسلة القوى**

A **power series** is an infinite series of the form $\sum_{n=0}^{\infty} a_n x^n$.

Theorem (6.2): If $0 \leq x < 1$, then $\sum_{n=1}^{\infty} x^{n-1} = \frac{1}{1-x}$.

Proof:

Let $< S_n >$ be the sequence of n^th partial sums of $\sum_{n=1}^{\infty} x^{n-1}$

$S_n = 1 + x + x^2 + \cdots + x^n$

$\Rightarrow \; x\,S_n = x + x^2 + x^3 + \cdots + x^n + x^{n+1}$

$\Rightarrow \; S_n - x\,S_n = 1 - x^{n+1}$

$\Rightarrow \; S_n(1-x) = 1 - x^{n+1}$

$\Rightarrow \; S_n = \frac{1-x^{n+1}}{1-x}$

$\Rightarrow \; \sum_{n=1}^{\infty} x^{n-1} = \lim\limits_{n\to\infty} S_n$

$= \lim\limits_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \frac{1}{1-x}$

Example (6.12): Discuss the convergence of the series $\sum_{n=1}^{\infty} \frac{n^2}{x^n}$, on $(0,\infty)$.

Solution:

$\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \sqrt[n]{\frac{n^2}{x^n}}$

$= \lim\limits_{n\to\infty} \frac{\sqrt[n]{n^2}}{x}$

$= \frac{1}{x} \lim\limits_{n\to\infty} e^{\frac{2\log n}{n}} = \frac{1}{x}\, e^0 = \frac{1}{x}$, for $x \in (0,1)$

Hence the series converges if $\frac{1}{x} < 1 \quad \Leftrightarrow \quad x > 1$

and diverges if $\frac{1}{x} > 1 \quad \Leftrightarrow \quad x < 1$

Example (6.13): Discuss the convergence of the series of functions $\sum_{n=1}^{\infty} \frac{e^{2n}}{(1+|x-1|)^n}$.

Solution:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{e^{2n}}{(1+|x-1|)^n}}$$

$$= \lim_{n\to\infty} \frac{e^2}{(1+|x-1|)} = \frac{e^2}{1+|x-1|} \quad ,$$

Hence the series converges if

$$\frac{e^2}{1+|x-1|} < 1$$

$$\Leftrightarrow \quad e^2 - 1 < |x-1|$$

$$\Leftrightarrow \qquad x < e^2 \qquad and \qquad 2 - e^2 > x$$

Otherwise its diverge.

Example (6.14): Discuss the convergence of the series of functions $\sum_{n=1}^{\infty} \frac{2^n}{n!} x^n$

Solution:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}}{(n+1)!} x^{n+1}}{\frac{2^n}{n!} x^n}\right|$$

$$= \lim_{n\to\infty} \frac{2|x|}{n+1} = 0 < 1 \ , \ \forall \ x \in R.$$

Hence the series converges.

Example (6.15): Find the interval of the convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Solution: $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}}\right|$

$$= \lim_{n\to\infty} \frac{\frac{|x|^{n+1}}{(n+1)!}}{\frac{|x|^n}{n!}}$$

$$= \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1 \ , \ \forall \ x \in R.$$

$\therefore$ The interval of convergence is R.

Example (6.16): Find the interval of the convergence of the series $\sum_{n=0}^{\infty} 2^n\, x^n$.

Solution:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2^{n+1}\, x^{n+1}}{2^n\, x^n}\right|$$

$$= \lim_{n\to\infty} 2\,|x| = 2|x|$$

The series is converges if $2|x| < 1 \Leftrightarrow -\frac{1}{2} < x < \frac{1}{2}$

$\therefore$ The interval of convergence is $\left(-\frac{1}{2}, \frac{1}{2}\right)$.
