Chapter Four
تـكـامـل لـيـبـيـك
Lebesgue Integral

Definition (4.1): (Lebesgue Partition) تـجـزئـة لـيـبـيـك

Let Ω be a measurable set in R. The **Lebesgue partition** P of an interval Ω is a family of a finite number of measurable subsets of Ω, $P = \{A_i : i = 1, \dots, n\}$, such that

(1) A_i is measurable set in Ω, $\forall\ i = 1,2, \dots, n$.

(2) $A_i \cap A_j = \emptyset,\ \forall\ i \neq j$.

(3) $\bigcup_{i=1}^{n} A_i = \Omega$.

The sets A_i , $i = 1,2, \dots, n$ are called **components** (مركبات) of P.

Definition (4.2): (Lower and Upper Lebesgue Sums) مجاميع ليبيك العليا والسفلى

Let f be a bounded function on Ω, and P a partition of Ω given by $P = \{A_i : i = 1,2, \dots, n\}$. We denote by m_i and M_i the quantities

$$m_i = inf\{f(x): x \in A_i\} \quad \text{and} \quad M_i = sup\{f(x): x \in A_i\}.$$

Then the corresponding **lower** and **upper Lebesgue sums** for f on Ω are

$L(f, P) = \sum_{i=1}^{n} m_i\ \mu(A_i)$ and $U(f, P) = \sum_{i=1}^{n} M_i\ \mu(A_i)$

Definition (4.3): (Refinement) الـتـنـعـيـم

Let $P = \{A_i : i = 1,2, \dots, n\}$ and $P' = \{B_j : j = 1,2, \dots, m\}$ are Lebesgue partitions of Ω, P' is a **refinement** of P, if $\forall\ j$, $\exists\ i$, such that $B_j \subseteq A_i$.

Definition (4.4): (Lebesgue Integrable) **قابلية التكامل الليبيكي**

Let f be a bounded function on a measurable set Ω. Then we define:

- The **lower Lebesgue integral of f** on Ω to be $\underline{\mathcal{L}}\int_{\Omega} f = \sup_{P} L(f,P)$,
- The **upper Lebesgue integral of f** on Ω to be $\overline{\mathcal{L}}\int_{\Omega} f = \inf_{P} U(f,P)$,

Where P denotes Lebesgue partition of Ω.

Further, if the lower and upper integrals are equal, we say that f is **Lebesgue-integrable** on Ω, $\int_{\Omega} f$ or $\mathcal{L}\int_{\Omega} f$ to be their common value; that is

$$\mathcal{L}\int_{\Omega} f = \overline{\mathcal{L}}\int_{\Omega} f = \overline{\mathcal{L}}\int_{\Omega} f$$

Theorem (4.1): If $f:[a,b] \to R$ is Riemann-integrable then f is Lebesgue-integrable and

$$\mathcal{L}\int_{[a,b]} f = \mathcal{R}\int_{a}^{b} f$$

Proof:

Since $[a,b]$ is a measurable set and every Riemann partition of $[a,b]$ is Lebesgue partition of $[a,b]$

Hence,

$\underline{\mathcal{R}}(f) \subseteq \underline{\mathcal{L}}(f)$ and $\overline{\mathcal{R}}(f) \subseteq \overline{\mathcal{L}}(f)$

$\Rightarrow \sup\{\underline{\mathcal{R}}(f)\} \leq \sup\{\underline{\mathcal{L}}(f)\}$

$$\Rightarrow \underline{\mathcal{R}}\int f \leq \underline{\mathcal{L}}\int f \qquad \ldots(1)$$

Also,

$\inf\{\overline{\mathcal{R}}(f)\} \leq \inf\{\overline{\mathcal{L}}(f)\}$

$$\Rightarrow \overline{\mathcal{R}}\int f \leq \overline{\mathcal{L}}\int f \qquad \dots(2)$$

From (1) and (2), we get

$$\underline{\mathcal{R}}\int f \leq \underline{\mathcal{L}}\int f \leq \overline{\mathcal{L}}\int f \leq \overline{\mathcal{R}}\int f \qquad \dots(3)$$

Since f is Riemann-integrable we have

$$\mathcal{R}\int f = \underline{\mathcal{R}}\int f = \overline{\mathcal{R}}\int f$$

Then from (3), we get

$$\mathcal{L}\int f = \underline{\mathcal{L}}\int f = \overline{\mathcal{L}}\int f$$

$$\therefore \ f \text{ is Lebesgue} - \text{integrable and } \ \mathcal{L}\int_{[a,b]} f = \mathcal{R}\int_a^b f$$

Note (4.1): The converse of the above theorem is not true, i.e.

If f is Lebesgue-integrable $\nRightarrow$ f is Riemann-integrable.

As shown in the following example.

Example (4.1): Let f be a function defined on [0,1] by

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1, \quad x \ \ rational \\ 0, & 0 \leq x \leq 1, \quad x \ \ irrational \end{cases}$$

Prove that f is Lebesgue-integrable but it is not Riemann-integrable.

Solution:

Let $P = \{A_1, A_2\}$ be a measurable partition of $[0, 1]$

Where

$A_1 = Q \subset [0,1]$

$A_2 = Q' \subset [0,1]$

$$L(f,P) = \sum_{i=1}^{2} m_i \, \mu(A_i)$$

$$= m_1 \, \mu(A_1) + m_2 \, \mu(A_2)$$

$$= 1 \times 0 + 0 \times 1 = 0$$

$$\Rightarrow \quad \underline{\mathcal{L}} \int_{[0,1]} f = 0$$

and

$$U(f,P) = \sum_{i=1}^{2} M_i \, \mu(A_i)$$

$$= M_1 \, \mu(A_1) + M_2 \, \mu(A_2)$$

$$= 1 \times 0 + 0 \times 1 = 0$$

$$\Rightarrow \quad \overline{\mathcal{L}} \int_{[0,1]} f = 0$$

Since $\underline{\mathcal{L}} \int_{[0,1]} f = \overline{\mathcal{L}} \int_{[0,1]} f$

$\Rightarrow$ f is Lebesgue-integrable on [0,1].

Now, to prove that f is not Riemann-integrable

Let $P = \{[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]\}$ be any partition of [0,1]. Then

$$L(f,P) = \sum_{i=1}^{n} m_i \delta x_i$$

$$= \sum_{i=1}^{n} 0 \times \delta x_i = 0$$

$$\Rightarrow \quad \int_{\underline{0}}^{1} f = 0$$

and

$$U(f,P) = \sum_{i=1}^{n} M_i \delta x_i$$

$$= \sum_{i=1}^{n} 1 \times \delta x_i$$

$$= \sum_{i=1}^{n} \delta x_i = 1$$

$$\Rightarrow \quad \overline{\int_0^1} f = 1$$

Since $\underline{\int_0^1} f \neq \overline{\int_0^1} f$

$\Rightarrow$ f is not Riemann-integrable on [0,1].

Theorem (4.2): (Lebesgue's Criterion for integrability)

Let f be a bounded function on a measurable bounded Ω on R. Then f is Lebesgue-integrable on Ω if and only if for each positive number ε, there is a Lebesgue partition P of Ω for which $U(f,P) - L(f,P) < \varepsilon$.

Theorem (4.3): Suppose that $f: \Omega \to R$ is a bounded Lebesgue-integrable on a bounded measurable set Ω. We set

$$m = \inf_{x \in \Omega} f(x), \qquad M = \sup_{x \in \Omega} f(x)$$

Then $\quad m\, \mu(\Omega) \leq \int_{\Omega} f(x)\, dx \leq M\, \mu(\Omega).$

Theorem (4.4): If f and g are Lebesgue-integrable, then $f+g$, fg and cf, for every constant c, are Lebesgue-integrable.

Theorem (4.5): Let Ω be a bounded measurable set in R. If $f: \Omega \to R$ is a bounded Lebesgue-integrable on Ω. Then $|f|$ is also Lebesgue-integrable on Ω and

$$\left| \int_{\Omega} f(x)\, dx \right| \leq \int_{\Omega} |f(x)|\, dx.$$

Exercises (4.1): (Homework)

(1) Let Ω be a bounded measurable set on R, and let $f: \Omega \to R$ be a function defined as $f(x) = a$, $\forall\, x \in \Omega$, (constant function). Prove that f is Lebesgue-integrable and

$$\int_{\Omega} f\ d\mu = a\, \mu(\Omega).$$

(2) Let $f: [a,b] \to R$ be a function defined as

$$f(x) = \begin{cases} -5, & if \quad x \in [a,b] \cap Q \\ \ \ 3, & if \quad x \in [a,b] \cap Q' \end{cases}$$

Show whether f is Riemann-integrable or Lebesgue-integrable.
