### Answers

**3.** $\dfrac{s}{1+s^2}$ **4.** $\dfrac{\pi}{2}$ if $0 < x < 1$ and 0 if $x > 1$ **5.** $f(x) = \dfrac{2(x - \sin x)}{\pi x^2}$

**6.** $f(x) = \dfrac{2(1 - \cos x)}{\pi x^2}$.

### Hint

**6.** We have $\displaystyle\int_0^{\infty} \frac{2(1-\cos x)}{\pi x^2} \cos \lambda x \, dx = \begin{cases} 1-\lambda, & if \quad 0 \le \lambda \le 1 \\ 0, & if \quad \lambda > 1 \end{cases}$.

Taking limits as $\lambda \to 0$, we get $\displaystyle\frac{2}{\pi}\int_0^{\infty} \frac{1-\cos x}{x^2} . 1 \, dx = 1$

$$\Rightarrow \qquad \frac{4}{\pi}\int_0^{\infty} \frac{\sin^2 \frac{x}{2}}{x^2} dx = 1$$

$$x/2 = t \quad \Rightarrow \quad dx = 2\,dt$$

$$\therefore \qquad \frac{4}{\pi}\int_0^{\infty} \frac{\sin^2 t}{4t^2} 2\,dt = 1.$$

## 5.8. LINEARITY OF TRANSFORMS

In this section, we shall prove that the Fourier transform, the Fourier sine and cosine transforms are linear operations.

**Theorem I.** *If f(x) and g(x) be any functions whose Fourier transforms exist then for any constants a and b then prove that*

$$F(af(x) + bg(x)) = aF(f(x)) + bF(g(x)).$$

**Proof.** By definition,

$$\begin{aligned} \mathrm{F}(af(x) + bg(x)) &= \int_{-\infty}^{\infty} (af(x) + bg(x))e^{-isx} \, dx \\ &= a\int_{-\infty}^{\infty} f(x)\, e^{-isx} \, dx + b\int_{-\infty}^{\infty} g(x)\, e^{-isx} \, dx \\ &= a\mathrm{F}(f(x)) + b\mathrm{F}(g(x)). \end{aligned}$$

**Theorem II.** *If f(x) and g(x) be any functions whose Fourier sine transforms exist then for any constants a and b prove that*

$$F_S(af(x) + bg(x)) = aF_S(f(x)) + bF_S(g(x)).$$

**Proof.** By definition,

$$\begin{aligned} \mathrm{F_S}\,(af(x) + bg(x)) &= \int_0^{\infty} (af(x) + bg(x)) \sin sx \, dx \\ &= a\int_0^{\infty} f(x) \sin sx \, dx + b\int_0^{\infty} g(x) \sin sx \, dx \\ &= a\mathrm{F_S}(f(x)) + b\mathrm{F_S}(g(x)). \end{aligned}$$

**Theorem III.** *If f(x) and g(x) be any functions whose Fourier cosine transforms exist then for any constants a and b then prove that*

$$F_C\,(af(x) + bg(x)) = aF_C(f(x)) + bF_C(g(x)).$$

**Proof.** By definition,

$$F_C(af(x) + bg(x)) = \int_0^{\infty} (af(x) + bg(x)) \cos sx \, dx$$
$$= a\int_0^{\infty} f(x) \cos sx \, dx + b\int_0^{\infty} g(x) \cos sx \, dx$$
$$= aF_C(f(x)) + bF_C(g(x)).$$

**Remark.** On the same lines, we can also prove that the inverse Fourier transform, the inverse Fourier sine and cosine transforms are also linear operations.

## 5.9. CHANGE OF SCALE PROPERTY OF TRANSFORMS

**Theorem I.** *If the Fourier transform of the function f(x) is* $\bar{f}(s)$ *then prove that the Fourier transform of f(ax) is* $\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$*, where a > 0.*

**Proof.** We have $\bar{f}(s) = F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$.

$$\therefore \quad F(f(ax)) = \int_{-\infty}^{\infty} f(ax)\, e^{-isx}\, dx$$

Let $t = ax$. $\therefore$ $dt = a dx$

$$\therefore \quad F(f(ax)) = \int_{-\infty}^{\infty} f(t)\, e^{-is(t/a)}\, \frac{dt}{a} \qquad (\because \quad a > 0)$$
$$= \frac{1}{a}\int_{-\infty}^{\infty} f(t)\, e^{-i(s/a)t}\, dt = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right).$$

**Theorem II.** *If the Fourier sine transform of the function f(x) is* $\bar{f}_S(s)$ *then prove that the Fourier sine transform of f(ax) is* $\frac{1}{a}\bar{f}_S\left(\frac{s}{a}\right)$*, where a > 0.*

**Proof.** We have $\bar{f}_S(s) = F_S(f(x)) = \int_0^{\infty} f(x) \sin sx \, dx$.

$$\therefore \quad F_S(f(ax)) = \int_0^{\infty} f(ax) \sin sx \, dx$$

Let $t = ax$. $\therefore$ $dt = a dx$

$$\therefore \quad F_S(f(ax)) = \int_0^{\infty} f(t) \sin(s(t/a))\, \frac{dt}{a} \qquad (\because \quad a > 0)$$
$$= \frac{1}{a}\int_0^{\infty} f(t) \sin((s/a)t)\, dt = \frac{1}{a}\bar{f}_S\left(\frac{s}{a}\right).$$

**Theorem III.** *If the Fourier cosine transform of the function f(x) is* $\bar{f}_C(s)$ *then prove that the Fourier cosine transform of f(ax) is* $\frac{1}{a}\bar{f}_C\left(\frac{s}{a}\right)$*, where a > 0.*

**Proof.** We have $\bar{f}_C(s) = F_C(f(x)) = \int_0^{\infty} f(x) \cos sx \, dx$.

$$\therefore \qquad F_C(f(ax)) = \int_0^\infty f(ax)\cos sx\, dx$$

Let $t = ax$. $\therefore$ $dt = a dx$

$$\therefore \qquad F_C(f(ax)) = \int_0^\infty f(t)\cos(s(t/a))\frac{dt}{a} \qquad (\because \quad a > 0)$$

$$= \frac{1}{a}\int_0^\infty f(t)\cos((s/a)t)\, dt = \frac{1}{a}\bar{f}_C\left(\frac{s}{a}\right).$$

## 5.10. TRANSFORMS OF DERIVATIVES

In this section, we shall derive the formulae to find the Fourier transform and the Fourier sine and cosine transforms of the derivatives of a function.

**Theorem I.** *If f(x) is any function for which the Fourier transforms of f(x) and f′(x) exist and f(x) → 0 as |x| → ∞ then prove that*

$$F(f'(x)) = is\, F(f(x)).$$

**Proof.** By definition,

$$F(f(x)) = \int_{-\infty}^\infty f(x)e^{-isx}\, dx \qquad ...(1)$$

and

$$F(f'(x)) = \int_{-\infty}^\infty f'(x)e^{-isx}\, dx \qquad ...(2)$$

$$(2) \Rightarrow \qquad F(f'(x)) = \int_{-\infty}^\infty \underset{\mathrm{I}}{e^{-isx}}\, \underset{\mathrm{II}}{f'(x)}\, dx$$

Integrating by parts, we get

$$F(f'(x)) = e^{-isx} f(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^\infty (-is)\, e^{-isx} f(x)\, dx$$

$$= 0 - 0 + is\int_{-\infty}^\infty f(x)\, e^{-isx}\, dx = is\, F(f(x)).$$

$$(\because \quad f(x) \to 0 \text{ as } |x| \to \infty)$$

**Remark.** By repeated application of the above theorem, we have

$$F(f''(x)) = is\, F(f'(x)) = (is).(is)F(f(x)) = -s^2F(f(x))$$

$\therefore$ $\mathbf{F(f''(x)) = -s^2F(f(x)).}$

By using **P.M.I.,** we can prove that

$$\mathbf{F(f^{(n)}(x)) = (is)^nF(f(x)),\ n \in N.}$$

**Theorem II.** *If f(x) is any function for which the Fourier sine and cosine transforms of f(x) and f′(x) exist and f(x) → 0 as x → ∞ then prove that*

*(i)* $F_S(f'(x)) = -sF_C(f(x))$ *(ii)* $F_C(f'(x)) = sF_S(f(x)) - f(0)$.

**Proof.** By definition $\qquad F_S(f(x)) = \int_0^\infty f(x)\sin sx\, dx$

and

$$F_C(f(x)) = \int_0^\infty f(x)\cos sx\, dx.$$

(*i*) $$F_S(f'(x)) = \int_0^\infty f'(x) \sin sx \, dx = \int_0^\infty \underset{\text{I}}{\sin sx} \, . \underset{\text{II}}{f'(x)} \, dx$$

$$= \sin sx \, f(x) \Big|_0^\infty - \int_0^\infty s \cos sx \, f(x) \, dx$$

$$= 0 - 0 - s\int_0^\infty f(x) \cos sx \, dx = - sF_C(f(x)).$$

$$(\because \quad f(x) \to \quad 0 \text{ as } x \to \infty)$$

(*ii*) $$F_C(f'(x)) = \int_0^\infty f'(x) \cos sx \, dx = \int_0^\infty \underset{\text{I}}{\cos sx} \, . \underset{\text{II}}{f'(x)} \, dx$$

$$= \cos sx \, f(x) \Big|_0^\infty - \int_0^\infty - s \sin sx \, f(x) \, dx$$

$$= 0 - (\cos 0) f(0) + s\int_0^\infty f(x) \sin sx \, dx \quad (\because \quad f(x) \to \quad 0 \text{ as } x \to \infty)$$

$$= - f(0) + sF_S(f(x)) = sF_S(f(x)) - f(0).$$

**Remarks.** (*i*) $F_S(f''(x)) = - sF_C(f'(x)) = - s[sF_S(f(x)) - f(0)] = - s^2F_S(f(x)) + sf(0).$

(*ii*) $F_C(f''(x)) = sF_S(f'(x)) - f'(0) = s[- sF_C(f(x))] - f'(0) = - s^2 F_C(f(x)) - f'(0).$

$\therefore$ $\mathbf{F_S(f''(x)) = - s^2 F_S(f(x)) + sf(0)}$ and $\mathbf{F_C(f''(x)) = - s^2 F_C(f(x)) - f'(0).}$

## ILLUSTRATIVE EXAMPLES

**Example 1.** *Find the Fourier sine and cosine transforms of the function $e^{-mx}$, $m > 0$ by using its second derivative.*

**Sol.** Let $f(x) = e^{-mx}$, $m > 0$.

$\therefore$ $f'(x) = - me^{-mx}$ and $f''(x) = (- m)(- m) e^{-mx} = m^2 e^{-mx}$

We have $F_S(f''(x)) = - s^2 F_S(f(x)) + sf(0).$

$\therefore$ $F_S(m^2 e^{-mx}) = - s^2 F_S(e^{-mx}) + se^0$

$\Rightarrow$ $m^2F_S(e^{-mx}) + s^2F_S(e^{-mx}) = s$

$\Rightarrow$ $(m^2 + s^2) F_S(e^{-mx}) = s \quad \Rightarrow \quad F_S(e^{-mx}) = \dfrac{\mathbf{s}}{\mathbf{m^2 + s^2}}.$

Also, $F_C(f''(x)) = - s^2 F_C(f(x)) - f'(0).$

$\Rightarrow$ $F_C(m^2e^{-mx}) = - s^2 F_C(e^{-mx}) - (- me^0)$

$\Rightarrow$ $m^2F_C(e^{-mx}) + s^2F_C(e^{-mx}) = m$

$\Rightarrow$ $(m^2 + s^2) F_C(e^{-mx}) = m \quad \Rightarrow \quad F_C(e^{-mx}) = \dfrac{\mathbf{m}}{\mathbf{m^2 + s^2}}.$

### 5.11. PARSEVAL'S IDENTITIES

Parseval's identities are for Fourier transforms and also for Fourier sine and cosine transforms. These identities are proved in the following theorems.

**Theorem I.** *If $\bar{f}(s)$ and $\bar{g}(s)$ are the Fourier transforms of the functions f(x) and g(x) respectively then prove that*

$$\int_{-\infty}^{\infty} f(x)\, g(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, \overline{\bar{g}(s)}\, ds \quad \textit{and} \quad \int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\bar{f}(s)|^2\, ds,$$

*where* $\overline{\bar{g}(s)}$ *represents the complex conjugate of the function* $\bar{g}(s)$.

**Proof.** $\displaystyle \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\,\overline{\bar{g}(s)}\, ds = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\left(\overline{\int_{-\infty}^{\infty} g(x)\, e^{-isx}\, dx}\right) ds$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\left(\int_{-\infty}^{\infty} \overline{g(x)\, e^{-isx}}\, dx\right) ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\left(\int_{-\infty}^{\infty} g(x)\, e^{isx}\, dx\right) ds \quad (\because \; g(x) \text{ is a real valued function})$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)\left(\int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx}\, ds\right) dx$$

(By changing the order of integration)

$$= \int_{-\infty}^{\infty} g(x)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx}\, ds\right) dx$$

$$= \int_{-\infty}^{\infty} g(x)\, f(x)\, dx = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$$

$$\therefore \quad \int_{-\infty}^{\infty} \mathbf{f(x)\, g(x)\, dx} = \frac{\mathbf{1}}{\mathbf{2\pi}}\int_{-\infty}^{\infty} \mathbf{\bar{f}(s)\, \overline{\bar{g}(s)}\, ds}. \qquad \text{...(1)}$$

In particular, let $g(x) = f(x)$.

$$\therefore \quad (1) \Rightarrow \quad \int_{-\infty}^{\infty} f(x)\, f(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\,\overline{\bar{f}(s)}\, ds.$$

$$\therefore \quad \int_{-\infty}^{\infty} |\mathbf{f(x)}|^2\, \mathbf{dx} = \frac{\mathbf{1}}{\mathbf{2\pi}}\int_{-\infty}^{\infty} |\mathbf{\bar{f}(s)}|^2\, \mathbf{ds}. \qquad (\because \; (f(x))^2 = |f(x)|^2)$$

**Theorem II.** *If* $\bar{f}_S(s)$ *and* $\bar{g}_S(s)$ *are the Fourier sine transforms of the functions f(x) and g(x) respectively then prove that*

$$\int_0^{\infty} f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^{\infty} \bar{f}_S(s)\, \bar{g}_S(s)\, ds \quad \textit{and} \quad \int_0^{\infty} (f(x))^2\, dx = \frac{2}{\pi}\int_0^{\infty} (\bar{f}_S(s))^2\, ds.$$

**Proof.** $\displaystyle \frac{2}{\pi}\int_0^{\infty} \bar{f}_S(s)\, \bar{g}_S(s)\, ds = \frac{2}{\pi}\int_0^{\infty} \bar{f}_S(s)\left(\int_0^{\infty} g(x) \sin sx\, dx\right) ds$

$$= \frac{2}{\pi}\int_0^{\infty} g(x)\left(\int_0^{\infty} \bar{f}_S(s) \sin sx\, ds\right) dx$$

(By changing the order of integration)

$$= \int_0^{\infty} g(x)\left(\frac{2}{\pi}\int_0^{\infty} \bar{f}_S(s) \sin sx\, ds\right) dx$$

$$= \int_0^{\infty} g(x)\, f(x)\, dx = \int_0^{\infty} f(x)\, g(x)\, dx$$

$$\therefore \quad \int_0^{\infty} \mathbf{f(x)\, g(x)\, dx} = \frac{\mathbf{2}}{\pi}\int_0^{\infty} \mathbf{\bar{f}_S(s)\, \bar{g}_S(s)\, ds}. \qquad \text{...(1)}$$

In particular, let $g(x) = f(x)$.

$$\therefore \quad \int_0^{\infty} \mathbf{(f(x))^2\, dx} = \frac{\mathbf{2}}{\pi}\int_0^{\infty} \mathbf{(\bar{f}_S(s))^2\, ds}.$$

**Theorem III.** *If $\bar{f}_C(s)$ and $\bar{g}_C(s)$ are the Fourier cosine transforms of the functions f(x) and g(x) respectively then prove that*

$$\int_0^\infty f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\bar{g}_C(s)\, ds \quad \text{and} \quad \int_0^\infty (f(x))^2\, dx = \frac{2}{\pi}\int_0^\infty (\bar{f}_C(s))^2\, ds.$$

**Proof.** $\dfrac{2}{\pi}\displaystyle\int_0^\infty \bar{f}_C(s)\,\bar{g}_C(s)\,ds = \frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\left(\int_0^\infty g(x)\cos sx\, dx\right)ds$

$$= \frac{2}{\pi}\int_0^\infty g(x)\left(\int_0^\infty \bar{f}_C(s)\cos sx\, ds\right)dx$$

(By changing the order of integration)

$$= \int_0^\infty g(x)\left(\frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\cos sx\, ds\right)dx$$

$$= \int_0^\infty g(x)\, f(x)\, dx = \int_0^\infty f(x)\, g(x)\, dx$$

$\therefore$ $$\int_0^\infty \mathbf{f(x)\, g(x)\, dx} = \frac{\mathbf{2}}{\pi}\int_0^\infty \mathbf{\bar{f}_C(s)\,\bar{g}_C(s)\, ds}. \quad \text{...(1)}$$

In particular, let $g(x) = f(x)$.

$\therefore$ (1) $\Rightarrow$ $$\int_0^\infty f(x)\, f(x)\, dx = \frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\,\bar{f}_C(s)\, ds$$

$\therefore$ $$\int_0^\infty \mathbf{(f(x))^2\, dx} = \frac{\mathbf{2}}{\pi}\int_0^\infty \mathbf{(\bar{f}_C(s))^2\, ds}.$$

**WORKING RULES FOR SOLVING PROBLEMS**

**Rule I.** (*i*) $F(f'(x)) = isF(f(x))$

(*ii*) $F(f^{(n)}(x)) = (is)^n F(f(x)),\ n \in \mathbf{N}$

**Rule II.** (*i*) $F_S(f'(x)) = -sF_C(f(x))$

(*ii*) $F_S(f''(x)) = -s^2F_S(f(x)) + sF(0)$

**Rule III.** (*i*) $F_C(f'(x)) = sF_S(f(x)) - f(0)$

(*ii*) $F_C(f''(x)) = -s^2F_C(f(x)) - f'(0)$

**Rule IV.** (*i*) $\displaystyle\int_{-\infty}^\infty f(x)\, g(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^\infty F(f(x))\,\overline{F(g(x))}\, ds$

(*ii*) $\displaystyle\int_{-\infty}^\infty |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^\infty |F(f(x))|^2\, ds$

**Rule V.** (*i*) $\displaystyle\int_0^\infty f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^\infty F_S(f(x))\, F_S(g(x))\, ds$

(*ii*) $\displaystyle\int_0^\infty (f(x))^2\, dx = \frac{2}{\pi}\int_0^\infty (F_S(f(x)))^2\, ds$

**Rule VI.** (*i*) $\displaystyle\int_0^\infty f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^\infty F_C(f(x))F_C(g(x))\, ds$

(*ii*) $\displaystyle\int_0^\infty (f(x))^2\, dx = \frac{2}{\pi}\int_0^\infty (F_C(f(x)))^2\, ds.$

## ILLUSTRATIVE EXAMPLES

**Example 1.** *If* $f(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$, *use the Parseval's identity for Fourier transform of* $f(x)$ *to show that* $\int_0^\infty \frac{\sin^2 ax}{x^2}\, dx = \frac{\pi a}{2}$.

**Sol.** We have $\qquad f(x) = \begin{cases} 1, & \text{if } |x| < a \\ 0, & \text{if } |x| > a \end{cases}$.

$$\therefore \qquad F(f(x)) = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{-a} 0 \,.\, e^{-isx}\, dx + \int_{-a}^{a} 1 \,.\, e^{-isx}\, dx + \int_{a}^{\infty} 0 \,.\, e^{-isx}\, dx$$

$$= 0 + \left. \frac{e^{-isx}}{-is} \right|_{-a}^{a} + 0$$

$$= -\frac{1}{is}(e^{-isa} - e^{isa}) = \frac{2}{s}\left(\frac{e^{isa} - e^{-isa}}{2i}\right) = \frac{2}{s} \sin sa$$

By Parseval's identity for Fourier transforms, we have

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(f(x))|^2\, ds\,.$$

$$\Rightarrow \quad \int_{-\infty}^{-a} (0)^2\, dx + \int_{-a}^{a} (1)^2\, dx + \int_{a}^{\infty} (0)^2\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left| \frac{2}{s} \sin sa \right|^2 ds$$

$$\Rightarrow \quad 0 + (a - (-a)) + 0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4\sin^2 as}{s^2}\, ds$$

$$\Rightarrow \quad \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2}\, dx = 2a \qquad \text{(Replacing } s \text{ by } x\text{)}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{\sin^2 ax}{x^2}\, dx = \pi a \quad \Rightarrow \quad 2\int_{0}^{\infty} \frac{\sin^2 ax}{x^2}\, dx = \pi a$$

$$\left( \because \frac{\sin^2 ax}{x^2} \text{ is an even function} \right)$$

$$\therefore \quad \int_{0}^{\infty} \frac{\sin^2 ax}{x^2}\, dx = \boldsymbol{\frac{\pi a}{2}}.$$

**Example 2.** *Using a Parseval's identity, show that :*

$$\int_0^\infty \frac{dx}{(a^2 + x^2)(b^2 + x^2)} = \frac{\pi}{2ab(a+b)}, \quad a, b > 0.$$

**Sol.** Let $\qquad f(x) = e^{-ax}$ and $g(x) = e^{-bx}$. $\qquad$ (Note this step)

$$\therefore \qquad \bar{f}_C(s) = \int_0^\infty e^{-ax} \cos sx\, dx = \frac{a}{a^2 + s^2}$$

and

$$\bar{g}_C(s) = \int_0^\infty e^{-bx} \cos sx\, dx = \frac{b}{b^2 + s^2}$$

$\therefore$ By Parseval's identity for Fourier cosine transforms, we have

$$\int_0^\infty f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\bar{g}_C(s)\, ds.$$

$$\Rightarrow \qquad \int_0^\infty e^{-ax}\, e^{-bx}\, dx = \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+s^2}\cdot\frac{b}{b^2+s^2}\, ds$$

$$\Rightarrow \qquad \int_0^\infty e^{-(a+b)x}\, dx = \frac{2ab}{\pi}\int_0^\infty \frac{ds}{(a^2+s^2)(b^2+s^2)}$$

$$\Rightarrow \qquad \frac{\pi}{2ab}\cdot\left.\frac{e^{-(a+b)x}}{-(a+b)}\right|_0^\infty = \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} \qquad \text{(Changing } s \text{ by } x\text{)}$$

$$\therefore \qquad \int_0^\infty \frac{dx}{(a^2+x^2)(b^2+x^2)} = -\frac{\pi}{2ab(a+b)}\ (0-1) = \boldsymbol{\frac{\pi}{2ab(a+b)}}.$$

**Example 3.** *Using a Parseval's identity, show that :*

$$\int_0^\infty \frac{\sin ax}{x(a^2+x^2)}\, dx = \frac{\pi}{2}\left(\frac{1-e^{-a^2}}{a^2}\right),\ a > 0.$$

**Sol.** Let $f(x) = e^{-ax},\ x > 0,\ a > 0$

and $$g(x) = \begin{cases} 1, & \text{if } \ 0 < x \le a \\ 0, & \text{if } \ x > a \end{cases}.$$

$$\therefore \qquad \bar{f}_C(s) = \int_0^\infty e^{-ax}\cos sx\, dx = \frac{a}{a^2+s^2}$$

and
$$\bar{g}_C(s) = \int_0^\infty g(x)\cos sx\, dx$$

$$= \int_0^a 1\,.\cos sx\, dx + \int_a^\infty 0\,.\cos sx\, dx$$

$$= \left.\frac{\sin sx}{s}\right|_0^a + 0 = \frac{1}{s}(\sin sa - \sin 0) = \frac{\sin as}{s}$$

$\therefore$ By Parseval's identity for Fourier cosine transforms, we have

$$\int_0^\infty f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^\infty \bar{f}_C(s)\,\bar{g}_C(s)\, ds\,.$$

$$\Rightarrow \qquad \int_0^a f(x)\, g(x)\, dx + \int_a^\infty f(x)\, g(x)\, dx = \frac{2}{\pi}\int_0^\infty \frac{a}{a^2+s^2}\cdot\frac{\sin as}{s}\, ds$$

$$\Rightarrow \qquad \int_0^a e^{-ax}\,.\,1\, dx + \int_a^\infty e^{-ax}\,.\,0\, dx = \frac{2a}{\pi}\int_0^\infty \frac{\sin as}{s(a^2+s^2)}\, ds$$

$$\Rightarrow \qquad \left.\frac{e^{-ax}}{-a}\right|_0^a + 0 = \frac{2a}{\pi}\int_0^\infty \frac{\sin ax}{x(a^2+x^2)}\, dx \qquad \text{(Replacing } s \text{ by } x\text{)}$$

$$\Rightarrow \qquad \frac{1}{-a}[e^{-a^2} - 1] = \frac{2a}{\pi}\int_0^\infty \frac{\sin ax}{x(a^2+x^2)}\, dx$$

$$\therefore \qquad \int_0^\infty \frac{\sin ax}{x(a^2+x^2)}\, dx = \frac{\pi}{\mathbf{2}}\left(\frac{\mathbf{1}-\mathbf{e}^{-\mathbf{a}^2}}{\mathbf{a}^2}\right).$$

## TEST YOUR KNOWLEDGE

**1.** Find the Fourier sine and cosine transforms of the function $e^{-3x/2}$ by using its second derivative.

**2.** Using a Parseval's identity, show that :

(*i*) $\displaystyle\int_0^\infty \frac{dx}{(x^2+4)(x^2+9)} = \frac{\pi}{60}$ (*ii*) $\displaystyle\int_{-\infty}^\infty \frac{dx}{(x^2+1)(x^2+16)} = \frac{\pi}{80}$.

**3.** Using a Parseval's identity, show that :

(*i*) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$ (*ii*) $\displaystyle\int_0^\infty \frac{x^2 dx}{(x^2+1)^2} = \frac{\pi}{4}$.

**4.** Use Parseval's identity for Fourier sine transform of the function $f(x) = \begin{cases} 1, & \text{if } 0 < x \le 1 \\ 0, & \text{if } x > 1 \end{cases}$ to show that $\displaystyle\int_0^\infty \left(\frac{1-\cos x}{x}\right)^2 dx = \frac{\pi}{2}$.

**5.** Use Parseval's identity for Fourier cosine transform of the function $f(x) = \begin{cases} 1, & \text{if } 0 < x \le 1 \\ 0, & \text{if } x > 1 \end{cases}$

to show that $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2} dx = \frac{\pi}{2}$.

### Answer

**1.** $\dfrac{4s}{9+4s^2}, \dfrac{6}{9+4s^2}$.

### Hint

**3.** (*ii*) Use Parseval's identity for the Fourier sine transform of the function $e^{-x}$, $x > 0$.