Triple Integrals in Spherical Coordinates
التكاملات الثلاثية بالإحداثيات الكروية

Example 1: Convert the point $(8,-\frac{\pi}{3},-\frac{\pi}{6})$ from spherical to rectangular coordinates.

Solution: $(\rho,\emptyset,\theta) \Longrightarrow (x,y,z)$

$x=\rho\ sin\emptyset\ \ \cos\theta \quad \Rightarrow x=8\sin(-\frac{\pi}{3})\ \ \cos(-\frac{\pi}{6})$

$\Rightarrow x=-8\sin(\frac{\pi}{3})\ \ \cos\left(\frac{\pi}{6}\right) \quad \Rightarrow x=-8\ \frac{\sqrt{3}}{2}\ \ \frac{\sqrt{3}}{2} \quad \Rightarrow \quad x=-6$

$y=\rho\ sin\emptyset\ \ \sin\theta \quad \Rightarrow y=8\sin(-\frac{\pi}{3})\ \ \sin(-\frac{\pi}{6})$

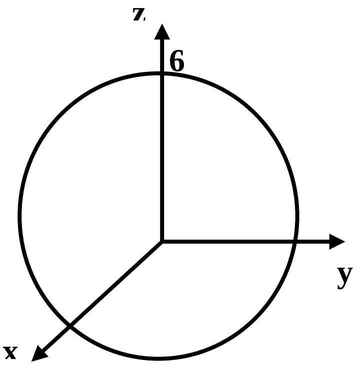
$\Rightarrow y=+8\sin(\frac{\pi}{3})\ \ \sin\left(\frac{\pi}{6}\right) \quad \Rightarrow x=8\ \frac{\sqrt{3}}{2}\ \ \frac{1}{2}\Rightarrow \quad y=2\sqrt{3}$

$z=\rho\ \ cos\emptyset \quad \Rightarrow z=8\ \ cos\left(-\frac{\pi}{3}\right)\Rightarrow z=8\ (\frac{1}{2})\ \Rightarrow \quad z=4$

$$\therefore (8,-\frac{\pi}{3},-\frac{\pi}{6}) \Longrightarrow (-6,\ \ 2\sqrt{3}\ \ ,4)$$

Example 2: By using the **spherical coordinate,** find the volume of the spherical with center origin point and radius equal to 6

Solution:



$$V=8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^6 \rho^2\ \ sin\emptyset\ \ d\rho\ \ \mathrm{d}\emptyset\ \mathrm{d}\theta=$$

$$8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\ \frac{\rho^3}{3}\Big]_0^6\ \ sin\emptyset\ \mathrm{d}\emptyset\ \mathrm{d}\theta=$$

$$8\frac{(6)^3}{3}\int_0^{\frac{\pi}{2}}-\cos\emptyset]_0^{\frac{\pi}{2}}\ \mathrm{d}\theta=\ 8\frac{(6)^3}{3}\int_0^{\frac{\pi}{2}}\ \mathrm{d}\theta=\ 8\frac{(6)^3}{3}\frac{\pi}{2}=4\frac{(6)^3\pi}{3}$$

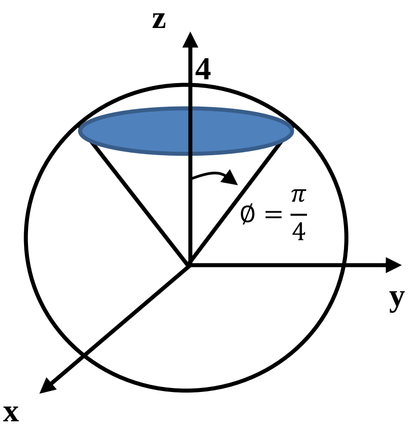
<u>Based on cylindrical coordinates</u>

$$8\int_0^{\frac{\pi}{2}}\int_0^{6}\int_0^{\sqrt{36-r^2}} dz\ \ \mathrm{rdr}\ \mathrm{d}\theta=8\int_0^{\frac{\pi}{2}}\int_0^{6}\ z]_0^{\sqrt{36-r^2}}\ \mathrm{rdr}\ \mathrm{d}\theta=8\int_0^{\frac{\pi}{2}}\int_0^{6}\ \sqrt{36-r^2}\ \mathrm{r\ dr\ d}\theta=$$

$$-4\int_0^{\frac{\pi}{2}}\frac{(36-r^2)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg]_0^6\ \ \mathrm{d}\theta=-\frac{8}{3}\int_0^{\frac{\pi}{2}}\left[(0)^{\frac{3}{2}}-(36)^{\frac{3}{2}}\right]\ \mathrm{d}\theta=\frac{8(6)^3}{3}\int_0^{\frac{\pi}{2}}\ \mathrm{d}\theta=4\frac{(6)^3\pi}{3}$$

Example 3: By using the **spherical coordinate,** find the solid volume above of the $z = \sqrt{x^2+y^2}$, below the sphere $x^2+y^2+z^2 = 16$

Solution: $z = \sqrt{x^2+y^2}$



$\rho\ cos\emptyset = \sqrt{\rho^2\ sin^2\emptyset\ \ \cos^2\theta + \rho^2\ sin^2\emptyset\ \ \sin^2\theta}$

$\rho\ cos\emptyset = \sqrt{\rho^2\ sin^2\emptyset[\cos^2\theta +\ \sin^2\theta\ ]}$

$\rho\ cos\emptyset = \sqrt{\rho^2\ sin^2\emptyset}$

$\rho\ cos\emptyset = \rho\ \ sin\ \emptyset\ \ \Rightarrow\ \ \frac{sin\ \emptyset}{cos\emptyset} = 1 \Rightarrow tan\emptyset = 1$

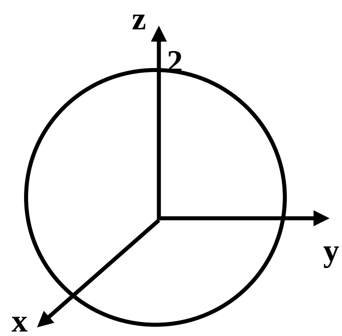
$\therefore \emptyset = \frac{\pi}{4}$

$$V = 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^{4}\rho^2\ \ sin\emptyset\ \ d\rho\ \mathrm{d}\emptyset\ \mathrm{d}\theta = 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\left.\frac{\rho^3}{3}\right]_0^4\ sin\emptyset\ \mathrm{d}\emptyset\ \mathrm{d}\theta =$$

$$= 4\frac{(4)^3}{3}\int_0^{\frac{\pi}{2}} \left.-\cos\emptyset\right]_0^{\frac{\pi}{4}}\ \mathrm{d}\theta = -\ 4\frac{(4)^3}{3}\int_0^{\frac{\pi}{2}}(\frac{1}{\sqrt{2}} - 1)\ \mathrm{d}\theta =$$

Example 4: Find the mas of the sphere with center origin points and radius equal 2, and the density at (x,y) is $\delta(x,y) = 1$.

Solution: $x^2+y^2+z^2 = 4$



1. Based on rectangular coordinates.

$$M = 8\int_0^{2}\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}}\delta\ dz\ dy\ dx =$$

2. Based on cylindrical coordinates.

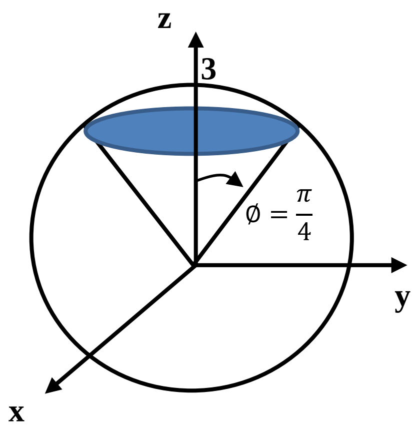
$$M = 8\int_0^{\frac{\pi}{2}}\int_0^{2}\int_0^{\sqrt{4-r^2}}\delta dz\ \ \mathrm{rdr\ d\theta} =$$

3. Based on spherical coordinates

$$M = 8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{2}\delta\ \rho^2\ \ sin\emptyset\ \ d\rho\ \ \mathrm{d}\emptyset\ \mathrm{d}\theta =$$

Example 5: By using the **spherical coordinate,** find the solid volume above of the cone (مخروط) $z^2 = x^2 + y^2$, below the sphere $x^2 + y^2 + z^2 = 9$

Solution:



$\rho^2 = x^2 + y^2 + z^2 = 9 \Rightarrow \quad \rho = 3$

The intersection of cone and sphere

$x^2 + y^2 + z^2 = 9$ ….(1)

$x^2 + y^2 = z^2$ …(2)

Substitute Eq.(2) in Eq. (1) we get

$z^2 + z^2 = 9 \Rightarrow \; 2z^2 = 9 \; \Rightarrow z = \frac{3}{\sqrt{2}}$

Since

$z = \rho \cos \emptyset \qquad \Rightarrow \frac{3}{\sqrt{2}} = 3 \cos \emptyset \Rightarrow \cos \emptyset \Rightarrow \frac{1}{\sqrt{2}}$

$$\therefore \emptyset = \frac{\pi}{4}$$

Or

$$z^2 = x^2 + y^2$$

$$\rho^2 \; cos^2\emptyset = \rho^2 \; sin^2\emptyset \; \cos^2\theta + \rho^2 \; sin^2\emptyset \; \sin^2\theta$$

$$\rho^2 \; cos^2\emptyset = \rho^2 \; sin^2\emptyset[\cos^2\theta + \sin^2\theta \,]$$

$$\rho^2 \; cos^2\emptyset = \rho^2 \; sin^2\emptyset$$

$$cos\emptyset = sin\emptyset$$

$$\frac{sin\,\emptyset}{cos\emptyset} = 1 \Rightarrow tan\emptyset = 1$$

$$\therefore \emptyset = \frac{\pi}{4}$$

$$V = 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^3 \rho^2 \; sin\emptyset \; d\rho \; \mathrm{d}\emptyset \, \mathrm{d}\theta = 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}} \left.\frac{\rho^3}{3}\right]_0^4 \; sin\emptyset \, \mathrm{d}\emptyset \, \mathrm{d}\theta =$$

$$= 4\frac{(4)^3}{3}\int_0^{\frac{\pi}{2}} -\cos\emptyset\Big]_0^{\frac{\pi}{4}} \mathrm{d}\theta = -4\frac{(4)^3}{3}\int_0^{\frac{\pi}{2}} \left(\frac{1}{\sqrt{2}} - 1\right) \mathrm{d}\theta =$$